

THE FIXED POINT PROPERTY IN c_0

Enrique Llorens-Fuster and Brailey Sims

1. Introduction

We say a closed convex subset of the Banach space $(X, \|\cdot\|)$ has the *fixed point property* (fpp) if every nonexpansive mapping $T : C \rightarrow C$ has a fixed point. Here, T nonexpansive means $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We ask which nonempty closed bounded convex subsets of c_0 enjoy the fpp?

It is now well known that all nonempty weak compact convex subsets of c_0 have the fpp [Maurey, 1980]. On the other hand, closed bounded convex subsets with a nonempty interior always fail to have the fpp, proposition 1 below. That sets without interior may also fail to have the fpp is demonstrated by $B_{c_0}^+ := \{(x_n) : 0 \leq x_n \leq 1, \text{ all } n\}$ on which $T : (x_n) \mapsto (1, x_1, x_2, \dots)$ is a fixed point free isometry.

We refine this last example by showing that closed bounded convex subsets of c_0 which are compact in a locally convex topology only ‘slightly’ coarser than the weak topology may fail to have the fpp. This lends support to the following.

Conjecture: In c_0 the only closed bounded convex subsets with the fpp are weak compact.

PROPOSITION 1: *Let C be a closed bounded convex subset of c_0 . If the set C has an interior point then C fails the fpp.*

PROOF: Without loss of generality we may suppose that $0 \in \text{int}(C)$, so there exists $\varepsilon > 0$ such that $B[0, \varepsilon] \subset C$.

We define $R : C \rightarrow B[0, \varepsilon]^+$ by

$$R((x(n))) = (|x(n)| \wedge \varepsilon)$$

where $|x(n)| \wedge \varepsilon := \min\{|x(n)|, \varepsilon\}$, and $B[0, \varepsilon]^+ = \{(x(n)) \in B[0, \varepsilon] : x(n) \geq 0\}$. In order to prove that R is nonexpansive, we apply the well known James-Birkhoff inequality:

$$|a \wedge \varepsilon - b \wedge \varepsilon| \leq |a - b|, \quad \text{for every } a, b, \varepsilon \in \mathbf{R}.$$

Therefore we have:

$$\begin{aligned} \|R(x) - R(y)\| &= \sup\{||x(n)| \wedge \varepsilon - |y(n)| \wedge \varepsilon| : n = 1, 2, \dots\} \\ &\leq \sup\{||x(n)| - |y(n)|| : n = 1, 2, \dots\} \leq \|x - y\|. \end{aligned}$$

Now we define the mappings $S : B[0, \varepsilon]^+ \rightarrow B[0, \varepsilon]^+$ by

$$S((x(n))) = (\varepsilon, x(1), x(2), \dots),$$

and $T : C \rightarrow B[0, \varepsilon]^+$ by $T := S \circ R$.

This map T is a nonexpansive selfmapping of C . If there exists $x \in C$ with $T(x) = x$, then $x \in B[0, \varepsilon]^+$, $R(x) = x$, and $T(x) = S(x) = x$, a contradiction. ■

2. The \mathcal{E} -topology on c_0 .

Let $d := (1, 1, 1, \dots, 1, \dots) \in \ell_\infty = c_0^{**}$, and let E be the closed subspace of ℓ_1 given by $E := \ker(d)$. That is, $E = \{(y(n)) \in \ell_1 : \sum y(n) = 0\}$. By [Guerre-Delabrière, 1992, lemma 1.1.11] E is a norming subspace for c_0 . Alternatively it is easily verified by direct calculation (see, for example, lemma 2.8 below) that in this case

$$\frac{1}{2}\|x\|_\infty \leq \sup\{\langle x, y \rangle : y \in E, \|y\|_1 \leq 1\} \leq \|x\|_\infty,$$

where $\langle x, y \rangle = \sum x(k)y(k)$, as usual. Consequently E separates points of c_0 and so, by [Jameson, 1974, 27.3], the set E is dense in $c_0^* = \ell_1$ with respect to the weak* topology. We consider c_0 equipped with the topology $\mathcal{E} := \sigma(c_0, E)$. That is, \mathcal{E} is the smallest locally convex linear topology on c_0 making continuous all the elements of E (as linear functionals on c_0).

The topology \mathcal{E} may be seen as ‘slightly’ coarser than the weak topology on c_0 . It displays some unusual, though not too pathological, properties. For example, the following five propositions can be proved by more or less standard methods of locally convex space theory.

PROPOSITION 2.1: *The topology \mathcal{E} consist of \emptyset , c_0 , all finite intersections of the sets*

$$\{(x(n)) \in c_0 : a < \sum x(n)y(n) < b, \sum y(n) = 0\}$$

and all arbitrary unions of these finite intersections.

PROPOSITION 2.2: *\mathcal{E} is Hausdorff.*

PROPOSITION 2.3: *A sequence (x_n) in c_0 is \mathcal{E} convergent to $x \in c_0$ if and only if for every $y \in E$,*

$$\langle x_n, y \rangle \longrightarrow \langle x, y \rangle .$$

PROPOSITION 2.4: *Every \mathcal{E} -convergent sequence is bounded.*

PROPOSITION 2.5: *Let M be a bounded subset of c_0 and let $x \in \mathcal{E}\text{-cl}M$. Then there exists a sequence (x_n) in M such that $x_n \xrightarrow{\mathcal{E}} x$.*

On the other hand, we have some results which are specific for the topology \mathcal{E} .

REMARK 2.6: The sequence (d_n) in c_0 given by

$$d_n := (\underbrace{1, \dots, 1}_n, 0, 0, \dots)$$

\mathcal{E} -converges to 0, but (d_n) does not have weakly null subsequences. Indeed, for $y = (y(n)) \in E$,

$$\langle d_n, y \rangle = \sum_{j=1}^n y(j) \longrightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Note that (d_n) is the standard *summing basis* for c_0 .

REMARK 2.7: Let (x_n) be a sequence in c_0 which is \mathcal{E} -convergent to $x \in c_0$. Since, the vector $y := \underbrace{(1, \dots, 1, -k, 0, \dots)}_k$ belongs to E , we have

$$x_n(1) + \dots + x_n(k) - kx_n(k+1) \longrightarrow x(1) + \dots + x(k) - kx(k+1)$$

and so

$$\frac{x_n(1) + \dots + x_n(k)}{k} - x_n(k+1) \longrightarrow \frac{x(1) + \dots + x(k)}{k} - x(k+1).$$

Necessary conditions such as this help provide a better understanding of \mathcal{E} -convergence.

LEMMA 2.8: For every element $x = (x(n)) \in c_0$ there exists a sequence (y_n) in E such that $\|y_n\|_1 = 2$ and

$$|\langle x, y_n \rangle| \longrightarrow \|x\|.$$

PROOF: Take $x(l) \in \{x(n) : n \in \mathbf{N}\}$ such that $|x(l)| = \|x\|$ and define

$$y_n := \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_l, \underbrace{(0, \dots, 0, -1, 0, \dots)}_n$$

Clearly $\|y_n\|_1 = 2$ and

$$|\langle x, y_n \rangle| = |x(l) - x(n+l)| \longrightarrow |x(l)| = \|x\|, \quad \text{as } n \rightarrow \infty.$$

■

PROPOSITION 2.9: If a sequence (x_n) in c_0 is \mathcal{E} -convergent to $x \in c_0$ then $\|x\| \leq 2 \liminf_n \|x_n\|$.

PROOF: Take $y \in E$. We have

$$|\langle x, y \rangle| = \lim |\langle x_n, y \rangle| \leq \|y\|_1 \liminf \|x_n\|$$

We now apply the above lemma, to obtain a sequence (y_n) in E with $\|y_n\|_1 = 2$ such that

$$|\langle x, y_n \rangle| \longrightarrow \|x\|, \quad \text{as } n \rightarrow \infty$$

and therefore the last inequality gives, for $n = 1, 2, \dots$

$$|\langle x, y_n \rangle| \leq \|y_n\|_1 \liminf_m \|x_m\|.$$

Taking limits we obtain the conclusion:

$$\|x\| = \lim_{n \rightarrow \infty} |\langle x, y_n \rangle| \leq 2 \liminf_m \|x_m\|$$

■

REMARK 2.10: The bound 2 in the last inequality cannot be improved. For example, if we consider the sequence $(d_n) \subset c_0$ defined above in remark 2.6, then for $e_1 := (1, 0, \dots)$ we have $d_n - 2e_1 \xrightarrow{\mathcal{E}} -2e_1$, but

$$\| -2e_1 \| = 2 = 2 \liminf \|d_n - 2e_1\|.$$

REMARK 2.11: There exist bounded, convex, norm-closed sets which are not \mathcal{E} -closed (That is, we do not have a Mazur's theorem for the \mathcal{E} -topology). To see this, let K be the norm closed convex hull of the set $D = \{d_n : n = 1, \dots\}$. Obviously every convex combination y of vectors d_n must verify $y(1) = 1$, and so $\|y\| = 1$. Therefore

$$0 = \mathcal{E} - \lim d_n \notin K,$$

and K is not \mathcal{E} -closed.

REMARK 2.12: The right shift $S : c_0 \rightarrow c_0$ is not \mathcal{E} -continuous. Indeed, the sequence (d_n) is \mathcal{E} -convergent to 0 but for $y \in E$ with $y(1) \neq 0$ we have

$$\langle S(d_n), y \rangle = \sum_{j=2}^n y(j) = \left(\sum_{j=1}^n y(j) \right) - y(1) \rightarrow -y(1), \quad \text{as } n \rightarrow \infty$$

and so $(S(d_n))$ does not converges to $S(0)$.

PROPOSITION 2.13: A sequence (x_n) in c_0 is weakly convergent to $x \in c_0$ if and only if $(S(x_n))$ is \mathcal{E} -convergent to $S(x)$.

PROOF: Since the right shift S is weak continuous we have that if $x_n \xrightarrow{w} x$ then $S(x_n) \xrightarrow{w} S(x)$, and so $S(x_n) \xrightarrow{\mathcal{E}} S(x)$. Conversely, for every $y = (y(1), y(2), \dots) \in \ell_1$ we have that

$$\tilde{y} := \left(-\sum y(j), y(1), y(2), \dots \right) \in E$$

If $S(x_n) \xrightarrow{\mathcal{E}} S(x)$ then $\langle S(x_n), \tilde{y} \rangle \rightarrow \langle S(x), \tilde{y} \rangle$. But it is easy to see that $\langle S(x_n), \tilde{y} \rangle = \langle x_n, y \rangle$ and $\langle S(x), \tilde{y} \rangle = \langle x, y \rangle$, which yields the conclusion. ■

\mathcal{E} -convergence can also be related to weak* convergence in $c_0^{**} = \ell_\infty$.

PROPOSITION 2.14: For a bounded sequence (x_n) in c_0 we have for the following conditions that (i) \implies (ii) \implies (iii).

- (i) For some λ_1 we have $\hat{x}_n \xrightarrow{w^*} \lambda_1 d$.
- (ii) $x_n \xrightarrow{\mathcal{E}} 0$.
- (iii) There exists a subsequence (x_{n_k}) with $\hat{x}_{n_k} \xrightarrow{w^*} \lambda_2 d$, for some $\lambda_2 \in \mathbf{R}$.

PROOF: If $\hat{x}_n \xrightarrow{w^*} \lambda_1 d$, then for $f \in \ker d$ we have $f(x_n) = \hat{x}_n(f) \rightarrow \lambda_1 d(f) = 0$, so (i) \implies (ii).

Suppose $x_n \xrightarrow{\mathcal{E}} 0$ and let $f_0 := (1/2, 1/4, 1/8, \dots, 1/2^n, \dots) \in \ell_1$, so $d(f_0) = 1$. Choose a subsequence x_{n_k} such that $\lim_k f_0(x_{n_k})$ exists, and equals λ_2 say. Then for $f \in c_0^* = \ell_1$ we have $f = d(f)f_0 + g$, where $g = f - d(f)f_0 \in E = \ker(d)$, and so $\hat{x}_{n_k}(f) = f(x_{n_k}) \rightarrow d(f)\lambda_2 = \lambda_2 d(f)$. Thus (ii) \implies (iii). ■

3. c_0 fails the \mathcal{E} -fpp

Let $d_0 := 0$ and for $n = 1, 2, 3, \dots$ define d_n as above;

$$d_n := (\underbrace{1, \dots, 1}_n, 0, 0, \dots)$$

To demonstrate the failure of the \mathcal{E} -fpp in c_0 , we show that

$$K := \overline{\text{co}} \{d_n\}_{n=0}^\infty$$

consisting of vectors of the form

$$\sum_{n=0}^\infty \lambda_n d_n = (1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), 1 - (\lambda_0 + \lambda_1 + \lambda_2), \dots),$$

where $\lambda_n \geq 0$ and $\sum_{n=0}^\infty \lambda_n = 1$, is a \mathcal{E} -compact convex set which admits a fixed point free affine isometry. Indeed T defined by

$$T(1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \dots) := (1, 1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \dots)$$

is such a map. The proof of these claims occupies the remainder of this section and is contained in the following lemmas.

LEMMA 3.1: *For the mapping T defined above we have*

- (i) T maps K into K ,
- (ii) T is an isometry,
- (iii) T is fixed point free in K .

PROOF: To establish (i) it suffices to note that for $\lambda_n \geq 0$ and $\sum_{n=0}^\infty \lambda_n = 1$, we have

$$\begin{aligned} T\left(\sum_{n=0}^\infty \lambda_n d_n\right) &= (1, 1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \dots) \\ &= \sum_{n=1}^\infty \lambda_{n-1} d_n \in K. \end{aligned}$$

(ii) follows, since for $x = (1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \dots)$ and $y = (1 - \mu_0, 1 - (\mu_0 + \mu_1), \dots)$ we have that

$$\begin{aligned} \|Tx - Ty\| &= \|(0, \mu_0 - \lambda_0, \mu_0 + \mu_1 - \lambda_0 - \lambda_1, \dots)\| \\ &= \|(\mu_0 - \lambda_0, \mu_0 + \mu_1 - \lambda_0 - \lambda_1, \dots)\| \\ &= \|x - y\|. \end{aligned}$$

Finally, if $x = (1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \dots)$ were such that $x = Tx = (1, 1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \dots)$ then we would have $\lambda_0 = 0, \lambda_1 = 0, \dots$ contradicting the requirement that $\sum_{n=0}^\infty \lambda_n = 1$. Indeed, $T(0) = (1, 0, 0, \dots) \neq 0$, and so we have (iii). ■

LEMMA 3.2: K is \mathcal{E} -closed.

PROOF: For $n = 1, 2, \dots$ let

$$x_n = \sum_{k=0}^{\infty} \lambda_k^{(n)} d_k = (1 - \lambda_0^{(n)}, 1 - \lambda_0^{(n)} - \lambda_1^{(n)}, \dots),$$

where $\lambda_k^{(n)} \geq 0$ and $\sum_{k=0}^{\infty} \lambda_k^{(n)} = 1$, be such that $x_n \xrightarrow{\mathcal{E}} x = (\mu_1, \mu_2, \dots)$.

Choosing $f := (1, -1, 0, 0, \dots) \in E$ we have

$$f(x_n - x) = (1 - \lambda_0^{(n)} - \mu_1) - (1 - \lambda_0^{(n)} - \lambda_1^{(n)} - \mu_2) \longrightarrow 0.$$

That is,

$$\lambda_1^{(n)} \longrightarrow \mu_1 - \mu_2.$$

Similarly, choosing $f := (0, 1, -1, 0, 0, \dots)$ we obtain

$$\lambda_2^{(n)} \longrightarrow \mu_2 - \mu_3,$$

and in general

$$\lambda_k^{(n)} \longrightarrow \mu_k - \mu_{k+1}.$$

Thus, for $k = 1, 2, \dots$

$$\lambda_k := \mu_k - \mu_{k+1} = \lim_n \lambda_k^{(n)} \geq 0$$

and

$$x = (\mu_1, \mu_1 - \lambda_1, \mu_1 - \lambda_1 - \lambda_2, \dots) \in c_0.$$

So we must have

$$\mu_1 = \sum_{k=1}^{\infty} \lambda_k \geq 0,$$

and then, provided $\mu_1 \leq 1$,

$$x = \sum_{k=1}^{\infty} \lambda_k d_k \in K$$

But, given $\epsilon > 0$ there exists N so that

$$\mu_1 = \sum_{k=1}^{\infty} \lambda_k < \sum_{k=1}^N \lambda_k + \epsilon/2,$$

and there exists n for which

$$|\lambda_k - \lambda_k^{(n)}| \leq \epsilon/2N, \quad \text{for } k = 1, 2, \dots, N.$$

Thus,

$$\mu_1 \leq \sum_{k=1}^N \lambda_k^{(n)} + \epsilon \leq 1 + \epsilon, \quad \text{as } \sum_{k=0}^{\infty} \lambda_k^{(n)} = 1,$$

and so $\mu_1 \leq 1$, as required. ■

Since $d_n \xrightarrow{\mathcal{E}} d_0$, we have that $\{d_n\}_{n=0}^\infty$ is \mathcal{E} -compact. The \mathcal{E} -compactness of K then follows from lemma 3.2, the definition of \mathcal{E} , and the following general result.

LEMMA 3.3: *Let X be a separable Banach space and let M be a closed norming subspace of X^* . If $D \subset X$ is $\sigma(X, M)$ -compact then $co(D)$ is $\sigma(X, M)$ -precompact.*

PROOF: Since M is closed and norming, D is bounded and equipped with the relative $\sigma(X, M)$ topology is a compact Hausdorff space. Let $\mathcal{C} := C(D, \sigma(X, M))$, the space of continuous real valued functions on D with this topology. Then V defined by

$$V(f)(m) := f(m|_D), \quad \text{for } f \in \mathcal{C}^* \text{ and } m \in M$$

is a weak* to weak*; that is, $\sigma(\mathcal{C}^*, \mathcal{C})$ to $\sigma(M^*, M)$, continuous linear operator from \mathcal{C}^* to M^* . Since M is norming, X may be identified with a closed subspace of M^* (the space $(X, \|\cdot\|')$ is complete, where $\|x\|' := \sup\{m(x) : m \in M, \|m\| \leq 1\}$). It suffices to show that $V(\mathcal{C}^*) \subseteq X$, as then $V(B_{\mathcal{C}^*})$ is a $\sigma(X, M)$ -compact convex subset of X containing D (for $d \in D$ consider the action of V on d regarded as a point measure in $B_{\mathcal{C}^*}$).

To establish that $V(\mathcal{C}^*) \subseteq X$ we first note that if $f \in \mathcal{C}^*$ then $V(f)$ is $\sigma(M, X)$ boundedly continuous. Indeed, since X is separable, bounded subsets of M are $\sigma(M, X)$ metrizable. So, if (m_n) is a bounded sequence in M with $m_n \rightarrow m$ in the $\sigma(M, X)$ topology then the Lebesgue dominated convergence theorem gives that $f(m_n|_D) \rightarrow f(m|_D)$, as required.

Now, suppose there is an $f \in \mathcal{C}^*$ with $g := V(f) \notin X$. Then there exists $F \in M^{**}$ with $\|F\| = 1$, $F(g) \neq 0$, and $F|_X = 0$. B_M is $\sigma(M^{**}, M^*)$ dense in $B_{M^{**}}$, so there is a net $(m_i) \subset B_M$ with $\hat{m}_i(m^*) \rightarrow F(m^*)$, for all $m^* \in M^*$. In particular $\hat{m}_i(x) \rightarrow F(x) = 0$, for all $x \in X \leq M^{**}$. That is, $m_i \rightarrow 0$ in the $\sigma(M, X)$ topology, and so since (m_i) is bounded $g(m_i) \rightarrow g(0) = 0$. But, $g \in M^*$ so $g(m_i) = \hat{m}_i(g) \rightarrow F(g) \neq 0$, a contradiction establishing the result. ■

4. Further results

In this section we note that the construction of the \mathcal{E} -topology can be generalized to obtain a family of similar topologies for some of which compact convex sets C may fail to have the fpp even for *contractive* mappings; that is, mappings $T : C \rightarrow C$ satisfying $\|Tx - Ty\| < \|x - y\|$, whenever $x \neq y$. Most of the proofs require only minor modifications to those given in sections 2 and 3 for the \mathcal{E} -topology, and so will be omitted.

To effect the generalization let $a = (a(n)) \in \ell_\infty$ be any sequence of ‘weights’ satisfying $\alpha \leq a(n) \leq \beta$, for some $0 < \alpha \leq \beta < \infty$, and take

$$\mathcal{E}_a := \sigma(c_0, \ker(a)),$$

the coarsest (locally convex linear topology) on c_0 making each functional in E_a continuous, where $E_a := \{y(n) \in \ell_\infty : \sum a(n)y(n) = 0\}$.

Proposition 2.1 remains true with the obvious modifications, namely:

PROPOSITION 4.1: *The topology \mathcal{E}_a consists of \emptyset , c_0 , all finite intersections of the sets*

$$\{(x(n)) \in c_0 : a < \sum x(n)y(n) < b, \sum a(n)y(n) = 0\}$$

and all arbitrary unions of these finite intersections.

Again E_a is a norming subspace for c_0 , indeed

$$\frac{\beta}{\alpha + \beta} \|x\|_\infty \leq \sup\{\langle x, y \rangle : y \in E_a, \|y\|_1 \leq 1\} \leq \|x\|_\infty,$$

so \mathcal{E}_a is Hausdorff.

Similarly one can verify propositions 2.3, 2.4 and 2.5 with E replaced by E_a and \mathcal{E} replaced by \mathcal{E}_a .

The sequence (d_n) need not converge to 0 with respect to the \mathcal{E}_a topology. Indeed, for $y = (y(n)) \in E_a$

$$\langle d_n, y \rangle = \sum_{j=1}^n y(j)$$

and it is generally untrue that the above sum converges to 0 as $n \rightarrow \infty$. On the other hand, if we replace (d_n) by the sequence (a_n) given by

$$a_n := (a(1), \dots, a(n), 0, 0, \dots)$$

we have

PROPOSITION 4.2: *The sequence a_n is \mathcal{E}_a -convergent to 0 and does not have weakly null subsequences. Indeed, for $y = (y(n)) \in E_a$,*

$$\langle a_n, y \rangle = \sum_{j=1}^n a(j)y(j) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Variants of lemma 2.8 and proposition 2.9 also hold for \mathcal{E}_a as do analogues of remarks 2.10, 2.11 and 2.12.

LEMMA 4.3: *For every element $x = (x(n)) \in c_0$ there exists (y_n) in E_a such that*

$$1 + \frac{\alpha}{\beta} \leq \|y_n\|_1 \leq 1 + \frac{\beta}{\alpha}$$

and

$$|\langle x, y_n \rangle| \rightarrow \|x\|.$$

The proof is essentially the same as that for lemma 2.8 if the -1 in the definition of y_n is replaced by $-a(l)/a(n+l)$.

Using this lemma we can prove the following in a way similar to that for proposition 2.9.

PROPOSITION 4.4: *If a sequence (x_n) in c_0 is \mathcal{E}_a -convergent to $x \in c_0$ then*

$$\|x\| \leq \left(1 + \frac{\beta}{\alpha}\right) \liminf_n \|x_n\|.$$

To obtain instances where the \mathcal{E}_a -fpp fails we put $a_0 := 0$ and take

$$K_a := \overline{\text{co}} \{a_n\}_{n=0}^\infty.$$

As we will see K_a consists of vectors of the form

$$\sum_{n=0}^\infty \lambda_n a_n = (a(1)(1 - \lambda_0), a(2)(1 - (\lambda_0 + \lambda_1)), a(3)(1 - (\lambda_0 + \lambda_1 + \lambda_2)), \dots),$$

where $\lambda_n \geq 0$ and $\sum_{n=0}^\infty \lambda_n = 1$.

That K_a is \mathcal{E}_a -closed follows by effectively the same argument as that used for lemma 3.2 with the functional f employed at the n 'th step of the induction replaced by $f := (0, \dots, 1, -a(n)/a(n+1), 0, \dots)$, where the 1 occurs in the n 'th position. This, in combination with proposition 4.2 and lemma 3.3, establishes the following.

PROPOSITION 4.5: *K_a is an \mathcal{E}_a -compact convex set.*

Now define T_a to be the affine map given by

$$T_a(a(1)(1 - \lambda_0), a(2)(1 - (\lambda_0 + \lambda_1)), \dots) := (a(1), a(2)(1 - \lambda_0), a(3)(1 - (\lambda_0 + \lambda_1)), \dots).$$

In other words,

$$T_a\left(\sum_{n=0}^\infty \lambda_n a_n\right) := \sum_{n=1}^\infty \lambda_{n-1} a_n.$$

It is clear that T_a maps K_a into K_a . Moreover, if $x = (a(1)(1 - \lambda_0), a(2)(1 - (\lambda_0 + \lambda_1)), \dots)$ were such that

$$x = T_a(x) = (a(1), a(2)(1 - \lambda_0), a(3)(1 - (\lambda_0 + \lambda_1)), \dots)$$

then we would have $\lambda_0 = 0, \lambda_1 = \lambda_0, \dots$ contradicting the requirement that $\sum_{n=0}^\infty \lambda_n = 1$, so T_a is fixed point free in K_a .

Further, if

$$x = (a(1)(1 - \lambda_0), a(2)(1 - (\lambda_0 + \lambda_1)), \dots)$$

and

$$y = (a(1)(1 - \mu_0), a(2)(1 - (\mu_0 + \mu_1)), \dots)$$

are elements of K_a then

$$\|x - y\| = \max\{a(1)|\mu_0 - \lambda_0|, a(2)|\mu_0 - \lambda_0 + \mu_1 - \lambda_1|, \dots\}.$$

On the other hand

$$Tx = (a(1), a(2)(1 - \lambda_0), a(3)(1 - (\lambda_0 + \lambda_1)), \dots),$$

$$Ty = (a(1), a(2)(1 - \mu_0), a(3)(1 - (\mu_0 + \mu_1)), \dots)$$

and so,

$$\|Tx - Ty\| = \max\{a(2)|\mu_0 - \lambda_0|, a(3)|\mu_0 - \lambda_0 + \mu_1 - \lambda_1|, \dots\}.$$

We therefore arrive at the following conclusion.

PROPOSITION 4.6: $T_a : K_a \longrightarrow K_a$ is a fixed point free (contractive) nonexpansive mapping of the nonempty \mathcal{E}_a -compact convex set K_a whenever the sequence of weights $a = (a_n)$ is (strictly) decreasing.

REMARK 4.7: Similar constructions and conclusions can be achieved in the James space J and in various equivalent renormings of c_0 . This leads us to ask the following.

QUESTION: To what extent can the above construction and results be extended

- (a) in c_0 , and
- (b) into other Banach spaces?

We also reiterate our earlier conjecture.

QUESTION: Dose the nonexpansive-fpp for a closed bounded convex set in c_0 characterize the set being weak compact?

ACKNOWLEDGEMENT: We wish to thank Michael Smyth for drawing to our attention the general form of lemma 3.3 and its proof.

References

- Guerre-Delabrière, S., 1992, *Classical sequences in Banach spaces*. Pure and Appl. Math, **166**, M. Dekker.
- Jameson, J.G.O., 1974, *Topology and Normed Spaces*, Chapman and Hall.
- Maurey, B., 1980, *Points fixes des contractions sur un convexe fermé de L^1* , Séminaire d'Analyse Fonctionnelle, 1980-81, Exposé num. VIII, Ecole Polytechnique, Palaiseau.

Departament d'Anàlisi Matemàtica
Facultat de Matemàtiques
Universitat de València
Doctor Moliner 50, 46100 Burjassot, Spain

Department of Mathematics
The University of Newcastle
NSW 2308, Australia