Clarke-Ledyaev Mean Value Inequalities in Smooth Banach Spaces

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Abstract. We refine and extend the Clarke-Ledyaev mean value inequality to smooth Banach spaces.

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1 Introduction

Recently, F. H. Clarke and Yu. Ledyaev proved a multidirectional mean value inequality [4, 5]. It turns out to be a powerful tool in many applications (see [5, 6]). This multidirectional mean value inequality was proved in two different settings: for (Fréchet) smooth functions in Banach spaces and for lower semicontinuous functions in Hilbert spaces. Using a method similar to [5] Clarke and Radulescu [8] extended the Hilbert space version of the mean value inequality to smooth Banach spaces for locally Lipschitz functions and observed that the results and proofs in [5] with small modifications are valid in uniformly smooth Banach spaces. In this note we give a complete extension of the Hilbert space result to smooth Banach spaces. Since many functional spaces encountered in application problems are not Hilbert spaces, this extension will enable us to apply this mean value inequality for many problems in a more general setting. In fact, we achieve a little more: we extend the mean value inequality to unbounded sets and we refine it so that it encompasses Zagrodny’s approximate mean value theorem [14] as an easy consequence. Our proof relies on a non-local fuzzy sum rule and is different from and somewhat simpler than the proof in [5].

In the remainder of this section we establish notation and preliminaries. We prove the nonlocal fuzzy sum rule in Section 2 and, then, use it to derive a refined version of the Clarke-Ledyaev multidirectional mean value inequality in smooth Banach spaces in Section 3.

Let $X$ be a Banach space with closed unit ball $B_X$ and with continuous real dual $X^*$. For a point $x \in X$ and a set $Y \subset X$, we denote $d(x,Y) := \inf\{\|x - y\| : y \in Y\}$ and $[x,Y] := \{x + t(y - x) : y \in Y, t \in [0,1]\}$. The diameter of points $x_n, n = 1, \ldots, N$ in $X$ is defined by $\text{diam}(x_1, \ldots, x_N) := \max\{\|x_n - x_m\| : n, m = 1, \ldots, N\}$. We use $\hat{R}$ to denote the extended real line $\hat{R} = \mathbb{R} \cup \{\infty\}$. Let us recall the definition of smooth subderivatives and normal cones.

**Definition 1.1** Let $f : X \to \hat{R}$ be a lower semicontinuous function and $S$ a closed subset of $X$. We say $f$ is $\beta$-subdifferentiable and $x^*$ is a $\beta$-subderivative of $f$ at $x$ if there exists a locally Lipschitz $\beta$-smooth function $g$ such that $\nabla g(x) = x^*$ and $f - g$ attains a local minimum at $x$. We denote the set of all $\beta$-subderivatives of $f$ at $x$ by $D_\beta f(x)$. We define the $\beta$-normal cone of $S$ at $x$ to be $N(x,S) := D_\beta \delta_S(x)$ where $\delta_S$ is the indicator function of $S$ defined by $\delta_S(x) = 0$ for $x \in S$ and $\infty$ otherwise.

In Definition 1.1 “$\beta$-smooth” represents various smoothness concepts of different strengths. Some well known examples are: Gateaux smooth ($\beta = G$): the function $g$ is Gateaux differentiable and $\nabla g$ is weak-star continuous in a neighborhood of $x$; Fréchet smooth ($\beta = F$): the function $g$ is Fréchet differentiable and $\nabla g$ is norm continuous in a neighborhood of $x$; $s$-Hölder smooth ($\beta = H(s), s \in (0,1]$): the function $g$ is $s$-Hölder differentiable and $\nabla g$ is norm continuous in a neighborhood of $x$. In a Hilbert space, $D_{H(1)}$ is the proximal subderivative, denoted by $\partial_e$ in [5].

In between the Gateaux and Fréchet smoothness there are also other useful smoothness concepts. The concept of bornology provides a uniform way of defining these concepts. A
bornology $\beta$ of $X$ is a family of closed bounded and centrally symmetric (convex) subsets of $X$ whose union is $X$, which is closed under multiplication by positive scalars and is directed upwards (that is, the union of any two members of $\beta$ is contained in some member of $\beta$). We will denote $X_\beta$ the dual space of $X$ endowed with the topology of uniform convergence on $\beta$-sets. If $\beta$ is a bornology we say a function $g$ is $\beta$-differentiable at $y$ and has a $\beta$-derivative $\nabla g(y)$ if

$$t^{-1}(g(y + tu) - g(y) - t(\nabla g(y), u)) \to 0$$

as $t \to 0$ uniformly in $u \in V$ for every $V \in \beta$. We say that a function $g$ is $\beta$-smooth at $x$ if $g$ is $\beta$-differentiable and $\nabla g : X \to X^*_\beta$ is continuous in a neighborhood of $x$. The most important bornologies are those formed by all (convex symmetric) bounded sets (Fréchet bornology denoted by $F$), weak compact sets (weak Hadamard bornology denoted by $WH$), compact sets (Hadamard bornology denoted by $H$) and finite dimensional sets (Gateaux bornology denoted by $G$). We refer to [2, 9, 13] for more details on these smoothness concept.

Another useful smoothness concept is when $X = \Pi_{k=1}^K X_k$ we say $g$ is $\beta = [\beta_1, ..., \beta_K]$-smooth provided that $g(x_1, x_2, ..., x_K) = \sum_{k=1}^K g_k(x_k)$ and, for each $k$, $g_k$ is $\beta_k$-smooth on $X_k$. This allows us to handle the product of several Banach spaces with different smoothness properties conveniently (see [15] for further discussion). By shifting a constant we may always assume that the $\beta$-smooth function $g$ in Definition 1.1 satisfies $g(x) = f(x)$.

We will need the following form of the Borwein-Preiss smooth variational principle [2] (see also [12]).

**Theorem 1.2** (Borwein-Preiss smooth variational principle) Let $X$ be a Banach space with an equivalent $\beta$-smooth norm, let $f : X \to \mathbb{R}$ be a lower semicontinuous function and let constants $\varepsilon > 0$ and $\lambda > 0$ be given. Suppose that $u$ satisfies

$$f(u) < \varepsilon + \inf_X f.$$ 

Then there exists a locally Lipschitz $\beta$-differentiable function $g$ on $X$ and $v$ in $X$ such that

(i) The function

$$x \to f(x) + g(x)$$

attains a global minimum at $x = v$, 

(ii) 

$$\|u - v\| < \lambda,$$

(iii) 

$$f(v) < \varepsilon + \inf_X f,$$

(iv) 

$$\|\nabla g(v)\| < \frac{2\varepsilon}{\lambda}. $$
2 A Fuzzy Sum Rule

In this section we prove a non-local fuzzy sum rule that will be used in the proof of the mean value inequality in the next section. It is also of independent interest.

**Theorem 2.1** Let $X$ be a Banach space with an equivalent $\beta$-smooth norm. Suppose that $f_1, \ldots, f_N : X \to \mathbb{R}$ are lower semicontinuous functions bounded below. Then, for any $\varepsilon > 0$, there exist $x_n, n = 1, \ldots, N$ and $x^*_n \in D_\beta f_n(x_n)$ satisfying $\text{diam}(x_1, \ldots, x_N) < \varepsilon$,

$$\text{diam}(x_1, \ldots, x_N) \cdot \max(\|x^*_1\|, \ldots, \|x^*_N\|) < \varepsilon,$$

and

$$\sum_{n=1}^N f_n(x_n) < \liminf_{\eta \to 0} \{ \sum_{n=1}^N f_n(x_n) \} \cdot \text{diam}(x_1, \ldots, x_N) \leq \eta \} + \varepsilon \quad (2)$$

such that

$$0 \in \sum_{n=1}^N x^*_n + \varepsilon B_{X^*}.$$

**Proof.** Define, for any real number $i > 0$,

$$w_i(y_1, \ldots, y_N) := \sum_{n=1}^N f_n(y_n) + i \sum_{n,m=1}^N \|y_n - y_m\|^2$$

and $M_i := \inf w_i$. Then $M_i$ is an increasing sequence and is bounded by $\lim_{\eta \to 0} \inf \{ \sum_{n=1}^N f_n(x_n) \} \cdot \text{diam}(x_1, \ldots, x_N) \leq \eta \}$. Let $M = \lim_{i \to \infty} M_i$. Observe that the product space of $N$ copies of $X$ (with the eucledian product norm) also has an equivalent $\beta$-smooth norm. For each $i$, applying the Borwein-Preiss smooth variational principle to function $w_i$, we obtain that there exist a $\beta$-smooth function $\phi_i$ and $x_{N,i}, n = 1, \ldots, N$ such that $w_i + \phi_i$ attains a local minimum at $(x_{1,i}, \ldots, x_{N,i})$, $\|\nabla \phi_i(x_{1,i}, \ldots, x_{N,i})\| < \varepsilon/N$ and

$$w_i(x_{1,i}, \ldots, x_{N,i}) \leq \inf w_i + \frac{1}{i} \leq M + \frac{1}{i}.$$

(3)

For each $n$, the function

$$y \to w_i(x_{1,i}, \ldots, x_{n-1,i}, y, x_{n+1,i}, \ldots, x_{N,i}) + \phi_i(x_{1,i}, \ldots, x_{n-1,i}, y, x_{n+1,i}, \ldots, x_{N,i})$$

attains a local minimum at $y = x_{n,i}$. Thus, for $n = 1, \ldots, N$, $x^*_{n,i} := -\nabla x_n \phi_i(x_{1,i}, \ldots, x_{N,i}) - 2i \sum_{m=1}^N \nabla \| (x_{n,i} - x_{m,i}) \| \in D_\beta f_n(x_{n,i})$. 

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Summing these $N$ inclusions leads to

$$
\sum_{n=1}^{N} x_{n,i} = -\sum_{n=1}^{N} \nabla x_n \phi_i(x_{1,i}, \ldots, x_{N,i}) - 2i \sum_{n=1}^{N} \sum_{m=1}^{N} \nabla \| x_{n,i} - x_{m,i} \|^2.
$$

Observing that $\| -\sum_{n=1}^{N} \nabla x_n \phi_i(x_{1,i}, \ldots, x_{N,i}) \| \leq \varepsilon$ and

$$
\nabla \| : \| x_{n,i} - x_{m,i} \|^2 + \nabla \| : \| x_{m,i} - x_{n,i} \|^2 = 0
$$

so that the double sum in the previous inclusion vanishes, we obtain

$$
0 \in \sum_{n=1}^{N} x_{n,i} + \varepsilon B_{X^*}.
$$

By the definition of $M_i$ we have

$$
M_{i/2} \leq w_{i/2}(x_{1,i}, \ldots, x_{N,i})
= w_i(x_{1,i}, \ldots, x_{N,i}) - \frac{i}{2} \sum_{n,m=1}^{N} \| x_{n,i} - x_{m,i} \|^2
\leq M_i + \frac{1}{i} - \frac{i}{2} \sum_{n,m=1}^{N} \| x_{n,i} - x_{m,i} \|^2.
$$

Rewriting (4) as

$$
\frac{i}{2} \sum_{n,m=1}^{N} \| x_{n,i} - x_{m,i} \|^2 \leq 2(M_i - M_{i/2} + \frac{1}{i})
$$

yields

$$
\lim_{i \to \infty} \frac{i}{2} \sum_{n,m=1}^{N} \| x_{n,i} - x_{m,i} \|^2 = 0.
$$

Therefore,

$$
\lim_{i \to \infty} \text{diam}(x_{1,i}, \ldots, x_{N,i}) = 0
$$

and

$$
\lim_{i \to \infty} \text{diam}(x_{1,i}, \ldots, x_{N,i}) \cdot \max(\| x_{1,i} \|, \ldots, \| x_{N,i} \|) = 0.
$$

Thus,

$$
M \leq \lim_{\eta \to 0} \inf \left\{ \sum_{n=1}^{N} f_n(x_n) : \text{diam}(x_1, \ldots, x_N) \leq \eta \right\}
\leq \lim_{i \to \infty} \inf \left\{ \sum_{n=1}^{N} f_n(x_{n,i}) = \lim_{i \to \infty} \inf \left\{ \sum_{n=1}^{N} f_n(x_{n,i}) \right\} \right\} \leq M
$$

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yields
\[ M = \liminf_{\eta \to 0} \left\{ \sum_{n=1}^{N} f_n(x_n) : \text{diam}(x_1, ..., x_N) \leq \eta \right\}. \]

It remains to take \( x_n = x_{n,i} \) and \( x_n^* = x_{n,i}^* \), \( n = 1, ..., N \) for a sufficiently large \( i \).

**Remark 2.2** (a) The fuzzy sum rule Theorem 2.1 is ‘nonlocal’ in the sense that it does not have any control on the locations of \( x_n, n = 1, ..., N \) unlike, for example, [3, Theorem 2.9]. (We only know that the points \( x_n, n = 1, ..., N \) are relatively close, i.e., \( \text{diam}(x_1, ..., x_N) < \varepsilon \).) The merit of this fuzzy sum rule is that it requires only a minimum assumption: the functions \( f_n^* \)s are lower semicontinuous. In [3, 10] (local) fuzzy sum rules were used to prove the uniqueness of viscosity solutions to certain Hamilton-Jacobi equations. The results in [3, 10] only apply to uniform continuous solutions because the (local) fuzzy sum rule requires a certain uniform continuity assumption. We can drop the uniform continuity requirement in those results by using Theorem 2.1 to replace the (local) fuzzy sum rule (e.g., use Theorem 2.1 to replace [3, Theorem 2.9] in the proof of [3, Theorem 3.2]). I am indebted to A. Święch and P. Wolenski for this observation.

(b) It is interesting that in some cases we can still gain control of the locations of \( x_n^* \)s by using the inequality (2). For example, let \( X \) be a Banach space with an equivalent \( \beta \)-smooth norm, let \( f : X \to \mathbb{R} \) be a lower semicontinuous function, let \( x \in \text{dom}(f) \) and let \( \varepsilon \in (0, 1) \). Applying Theorem 2.1 to \( f_1 = f + \delta_{x+B_X} \) and \( f_2 = \delta_{\{x\}} \) yields that there exist \( x_1 \) and \( x_2 \) such that \( \|x_1 - x_2\| < \varepsilon \), \( 0 \in D_\beta f_1(x_1) + D_\beta \delta_{\{x\}}(x_2) + \varepsilon B_X \), and

\[ f_1(x_1) + \delta_{\{x\}}(x_2) < f(x) + \varepsilon. \]

The last inequality forces \( x_2 = x \) and, therefore, \( x_1 \) must be in the interior of \( x + B_X \) so that \( D_\beta f_1(x_1) = D_\beta f(x_1) \). This leads to the well known conclusion that \( \text{dom}(D_\beta f) \) is dense in \( \text{dom}(f) \).

## 3 Clarke-Ledyaev mean value inequality

In this section, we refine and extend the Clarke-Ledyaev mean value inequality to Banach spaces with an equivalent \( \beta \)-smooth norm. We prove the following version of the mean value theorem first and then use it to extend the usual form of the mean value inequality and to deduce Zagrodny’s approximate mean value theorem.

**Theorem 3.1** Let \( X \) be a Banach space with an equivalent \( \beta \)-smooth norm, let \( Y \) be a nonempty, closed and convex subset of \( X \) and \( x \in X \) and let \( f : X \to \mathbb{R} \) be a lower semicontinuous function. Suppose that, for some \( h > 0 \), \( f \) is bounded below on \( [x, Y] + hB_X \) and

\[ \lim_{\eta \to 0} \inf_{y \in [x, Y] + \eta B_X} f(y) > f(x). \]

Then, there exists \( \bar{h} > 0 \) such that, for any \( \varepsilon > 0 \), there exist \( u \in [x, Y], d(u, Y) > \bar{h} \), \( z \in u + \varepsilon B_X \) and \( z^* \in D_\beta f(z) \) such that
(i) 
\[ \|z^*\| \cdot \|u - z\| < \varepsilon \]

(ii) 
\[ 0 \leq \langle z^*, w - u \rangle + \varepsilon \|w - u\| \text{ for all } w \in [x, Y]. \]

(iii) 
\[ 0 < \langle z^*, y - x \rangle + \varepsilon \|y - x\| \text{ for all } y \in Y. \]

Further, we can choose \( z \) to satisfy

(iv) 
\[ f(z) < \lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f + \varepsilon. \]

**Proof.** Let \( \tilde{f} = f + \delta_{[x,Y]} + hB_X \). Then \( \tilde{f} \) is bounded below on \( X \). Fix a \( \tilde{h} \in (0, h/2) \) such that \( \inf_{y \in Y + 2\tilde{h}B_X} f(y) > f(x) \). Without loss of generality we may assume that
\[ \varepsilon < \min \{ \inf_{y \in Y + 2\tilde{h}B_X} f(y) - f(x), \tilde{h} \}. \]

Applying the fuzzy sum rule Theorem 2.1 to \( f_1 = \tilde{f} \) and \( f_2 = \delta_{[x,Y]} \) we obtain that there exist \( z, u \) with \( \|z - u\| < \varepsilon \), \( z^* \in D_\beta \tilde{f}(z) = D_\beta f(z) \) and \( u^* \in N_\beta(u, [x,Y]) \) satisfying
\[ \max(\|z^*\|, \|u^*\|) \cdot \|z - u\| < \varepsilon \]
and
\[ f(z) < \lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f + \varepsilon \leq f(x) + \varepsilon \]
such that
\[ \|z^* + u^*\| < \varepsilon. \]

It remains to show that \( z^*, u \) and \( z \) satisfy (ii) and (iii). Since \([x,Y]\) is convex \( N_\beta(u, [x,Y]) \) coincides with the normal cone of \([x,Y]\) at \( u \) in the sense of convex analysis. Thus, \( u^* \in N_\beta(u, [x,Y]) \) implies that
\[ \langle u^*, w - u \rangle \leq 0, \quad \forall w \in [x,Y]. \]

Combining (7) and (8) yields
\[ 0 < \langle z^*, w - u \rangle + \varepsilon \|w - u\|, \quad \forall w \in [x,Y] \setminus \{u\}. \]

This implies (ii). Moreover, we must have \( d(u, Y) \geq \tilde{h} \) for otherwise we would have \( d(z, Y) \leq 2\tilde{h} \) and \( f(z) \geq \inf_{y \in Y + 2\tilde{h}B_X} f(y) > f(x) + \varepsilon \) which contradicts (6). Let \( u = x + \tilde{r}(y - x) \).

Then \( \tilde{h} \leq \|u - y\| = (1 - \tilde{r}) \|x - y\| \) implies \( 1 - \tilde{r} > 0 \). Clearly \( x \not\in Y \). For any \( y \in Y \) set \( w = y + \tilde{r}(y - y) \neq u \) in (9) yields (iii). This completes the proof. \( \blacksquare \)

Now we show that an extension of the usual form of the Clarke-Ledyaev mean value inequality follows easily from (less general looking) Theorem 3.1.
Theorem 3.2 Let $X$ be a Banach space with an equivalent $\beta$-smooth norm, let $Y$ be a nonempty, closed and convex subset of $X$ and $x \in X$ and let $f : X \to \mathbb{R}$ be a lower semicontinuous function. Suppose that, for some $h > 0$, $f$ is bounded below on $[x, Y] + hB_X$ and
\[
\lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f(y) - f(x) > r.
\]
Then, for any $\varepsilon > 0$, there exist $z \in [x, Y] + \varepsilon B$ and $z^* \in D_\beta f(z)$ such that
\[
r < \langle z^*, y - x \rangle + \varepsilon\|y - x\| \quad \text{for all } y \in Y.
\]
Further, we can choose $z$ to satisfy
\[
f(z) < \lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f + |r| + \varepsilon.
\]

Proof. Without loss of generality assume that the norm $\| \cdot \|$ of $X$ is $\beta$-smooth. Consider $X \times R$ with the ($\beta$-smooth) euclidean product norm. Take an $\varepsilon' \in (0, \varepsilon/2)$ small enough so that
\[
\lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f(y) - f(x) > r + \varepsilon'
\]
and define $F(z, t) := f(z) - (r + \varepsilon')t$. Obviously $F$ is lower semicontinuous on $X \times R$ and is bounded below on $[(x, 0), Y \times \{1\}] + hB_{X \times R}$. Moreover,
\[
\lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} F = \lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f - (r + \varepsilon') > f(x) = F(x, 0).
\]
Applying Theorem 3.1 with $f$, $x$ and $Y$ replaced by $F$, $(x, 0)$ and $Y \times \{1\}$ yields that there exist $(z, s) \in [(x, 0), Y \times \{1\}] + \varepsilon B_{X \times R}$, $z^* \in D_\beta f(z)$ satisfying
\[
f(z) - (r + \varepsilon')s < \lim_{\eta \to 0} \inf_{(w, t) \in [(x, 0), Y \times \{1\}] + \eta B_{X \times R}} (f(w) - (r + \varepsilon') t) + \varepsilon'
\]
i.e.,
\[
f(z) < \lim_{\eta \to 0} \inf_{(w, t) \in [(x, 0), Y \times \{1\}] + \eta B_{X \times R}} (f(w) - (r + \varepsilon')(t - s)) + \varepsilon'
\]
\[
\leq \lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f + |r| + \varepsilon
\]
such that, for all $y \in Y$,
\[
0 < \langle z^*, y - x \rangle - (r + \varepsilon') + \varepsilon' \sqrt{\|y - x\|^2 + 1}
\]
\[
\leq \langle z^*, y - x \rangle - r + \varepsilon'\|y - x\|
\]
\[
\leq \langle z^*, y - x \rangle - r + \varepsilon\|y - x\|.
\]
This completes the proof.

Setting $Y = X$ in Theorem 3.2 yields
Corollary 3.3 Let \( X \) be a Banach space with an equivalent \( \beta \)-smooth norm and let \( f : X \to \bar{R} \) be a lower semicontinuous function bounded below. Then, for any \( \varepsilon > 0 \), there exists \( z \in X \) such that \( d(0, D\beta f(z)) \leq \varepsilon \).

Remark 3.4 The term \( \varepsilon \| y - x \| \) in Theorem 3.2 cannot be dispensed with. Otherwise we would be able to replace the \( \varepsilon \) in the conclusion of Corollary 3.3 by 0 which is impossible, for example, for the function \( e^x : R \to R \). When \( Y \) is bounded, it is easy to see that \( \varepsilon \| y - x \| \) in Theorem 3.2 is redundant and we have:

Theorem 3.5 Let \( X \) be a Banach space with a \( \beta \)-smooth equivalent norm. let \( Y \) be a nonempty, closed, bounded and convex subset of \( X \) and \( x \in X \) and let \( f : X \to \bar{R} \) be a lower semicontinuous function. Suppose that, for some \( h > 0 \), \( f \) is bounded below on \([x, Y] + hB_X\) and

\[
\lim_{n \to 0} \inf_{y \in Y + \eta B_X} f(y) - f(x) > r.
\]

Then, for any \( \varepsilon > 0 \), there exist \( z \in [x, Y] + \varepsilon B_X \) and \( z^* \in D\beta f(z) \) such that

\[
r < \langle z^*, y - x \rangle \text{ for all } y \in Y.
\]

Further, we can choose \( z \) to satisfy

\[
f(z) < \lim_{n \to 0} \inf_{[x, Y] + \eta B_X} f + |r| + \varepsilon.
\]

In a Hilbert space we can take \( \beta = H(1) \). Then Theorem 3.5 recovers [5, Theorem 2.1]. (Note that \( \lim_{\eta \to 0} \inf_{[x, Y] + \eta B_X} f \leq \inf_{[x, Y]} f \).)

Remark 3.6 (a) In [7] the mean value inequality is used to prove a (local) fuzzy sum rule. The same method can be used to show that Theorem 3.2 is equivalent to Theorem 2.1 without the estimate (1). In fact, from the proof of Theorem 3.1 and Theorem 3.2 we can see that estimate (1) in Theorem 2.1 is used only for showing (i) in Theorem 3.1 which is not used in the proof of Theorem 3.2. Therefore, Theorem 2.1 without the estimate (1) implies Theorem 3.2. To show the converse, let \( y = (y_1, ..., y_N) \in X^N \), \( f := f_1(y_1) + f_2(y_2) + ... + f_N(y_N) : X^N \to \bar{R} \) and \( Y := \{(y, y, ..., y) \in X^N\} \) where \( X^N \) is the product space of \( N \) copies of \( X \) with the euclidean product norm. Choose an \( x = (\bar{x}_1, ..., \bar{x}_N) \in Y + \varepsilon' B_{X^N} \) such that

\[
\lim_{n \to 0} \inf_{y \in Y + \eta B_{X^N}} f - f(x) > -\varepsilon'.
\]

Then \([x, Y] \subset Y + \varepsilon' B_{X^N} \). Applying Theorem 3.2 yields that there exist \( z = (x_1, x_2, ..., x_N) \in Y + 2\varepsilon' B_{X^N} \), i.e., \( \text{diam}(x_1, ..., x_N) < 4\varepsilon' \) and \( z^* = (x_1^*, ..., x_N^*) \in D\beta f(z) \), i.e., \( x_n^* \in D\beta f_n(x_n) \) satisfying

\[
\sum_{n=1}^{N} f_n(x_n) < \lim_{\eta \to 0} \inf \sum_{n=1}^{N} f_n(y_n) \text{ : diam}(y_1, ..., y_N) \leq \eta \} + 2\varepsilon'
\]
such that
\[-\varepsilon' - \varepsilon' \|y - x\|_{X^*} < \langle z^*, y - x \rangle, \quad \forall y \in Y.\]  

(10)

We can rewrite (10) as
\[-\varepsilon' - \varepsilon' \sqrt{\sum_{n=1}^{N} \|u - \bar{x}_n\|^2} < \sum_{n=1}^{N} \langle x^*_n, u - \bar{x}_n \rangle, \quad \forall u \in X\]

or
\[-\varepsilon' - \varepsilon' \sqrt{\sum_{n=1}^{N} \|w + \bar{x}_1 - \bar{x}_n\|^2} - \sum_{n=2}^{N} \langle x^*_n, \bar{x}_1 - \bar{x}_n \rangle < \sum_{n=1}^{N} \langle x^*_n, w \rangle, \quad \forall w \in X.\]

It follows that
\[\| \sum_{n=1}^{N} x^*_n \| < 2\varepsilon' \sqrt{N}.

Taking \(\varepsilon' < \varepsilon \min(1/2\sqrt{N}, 1/4)\) we obtain Theorem 2.1 (without the estimate (1)).

(b) In [11], A. S. Lewis and D. Ralph established another interesting relation: the finite dimensional version of the mean value inequality (see [4]) is equivalent to a nonlinear duality theorem.

Now we use Theorem 3.1 to deduce Zagrodný’s approximate mean value theorem [14] as extended to allow \(f(y) = \infty\) in [1].

**Theorem 3.7** Let \(X\) be a Banach space with an equivalent \(\beta\)-smooth norm, let \(x, y \in X\) be two distinct points and let \(f : X \to \mathbb{R}\) be a lower semicontinuous function bounded below. Suppose that \(f(x) < \infty\) and \(f(y) - f(x) \geq r \in \mathbb{R}\). Then there exist a point \(c \in [x, y]\), sequences \(z_n\) and \(z^*_n \in D_\beta f(z_n)\) such that \((z_n, f(z_n)) \to (c, f(c))\) and

(i) \[0 \leq \lim \inf \langle z^*_n, c - z_n \rangle\]

(ii) \[r \leq \lim \inf \langle z^*_n, y - x \rangle.\]

**Proof.** We will consider the product space \(X \times \mathbb{R}\) with the euclidean product norm \(\| \cdot \|_{X \times \mathbb{R}}\). For each integer \(n\), applying Theorem 3.1 to \(F(w, v) = f(w) - (r - \frac{1}{n}) v, (x, 0)\) and \(Y = \{(y, 1)\}\) yields that there exist \((u_n, s_n) \in [[x, 0), (y, 1)], (z_n, t_n) \in (u_n, s_n) + \frac{1}{n}B_{(x, 0), (y, 1)}\) and \(z^*_n \in D_\beta f(z_n)\) such that
\[\| z^* \| : \| u_n - z_n \| < \frac{1}{n}, \]  

\[0 \leq \langle z^*, w - u_n \rangle - (r - \frac{1}{n}) (v - s_n) + \frac{1}{n} \| (w, v) - (u_n, s_n) \| \quad \forall (w, v) \in [[x, 0), (y, 1)], \]  

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0 \leq \langle z^*, y - x \rangle - (r - \frac{1}{n}) + \frac{1}{n} \|(y, 1) - (x, 0)\|, \quad (13)

and

\begin{align*}
  f(z_n) - (r - \frac{1}{n}) t_n &< \inf_{(w,v) \in [(x,0),(y,1)]} (f(w) - (r - \frac{1}{n})v) + \frac{1}{n} \\
  &\leq \inf_{(w,v) \in [(x,0),(y,1)]} (f(w) - rv) + \frac{2}{n}, \quad (14)
\end{align*}

Since \([x,0),(y,1)]\) is compact, passing to subsequences if necessary we may assume that both sequences \((u_n, s_n)\) and \((z_n, t_n)\) converge to a common limit \((c, \gamma) \in [(x,0),(y,1)]\). Then the lower semicontinuity of \(f\) and (14) implies that \(f(z_n) \rightarrow f(c)\). Setting \((w, v) = (c, \gamma)\) in (12) and in considering (11) we obtain

\begin{align*}
0 \leq \langle z^*, c - z_n \rangle + \frac{1}{n} - (r - \frac{1}{n})(\gamma - s_n) + \frac{1}{n} \|(c, \gamma) - (u_n, s_n)\|, \quad (15)
\end{align*}

Taking \(\liminf\) as \(n \to \infty\) in (13) and (15) completes the proof. \(\blacksquare\)

We conclude by remarking that the discussion in [3, Remark 2.14] applies to the results in this note. Therefore, in all the previous theorems the assumption "\(X\) is a Banach space with an equivalent \(\beta\)-smooth norm" can be replaced by the weaker assumption "\(X\) is a Banach space that admits a Lipschitz \(\beta\)-smooth bump function".

References


