Convergence of Lipschitz Regularizations of Convex Functions

JONATHAN M. BORWEIN* AND JON D. VANDERWERFF†

Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia V5A 1S6, Canada

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For a sequence or net of convex functions on a Banach space, we study pointwise convergence of their Lipschitz regularizations and convergence of their epigraphs. The Lipschitz regularizations we will consider are the infimal convolutions of the functions with appropriate multiples of the norm. For a sequence of convex functions on a separable Banach space we show that both pointwise convergence of their Lipschitz regularizations and Wijsman convergence of their epigraphs are equivalent to variants of two conditions used by Attouch and Beer to characterize slice convergence. Results for nonseparable spaces are obtained by separable reduction arguments. As a by-product, slice convergence for an arbitrary net of convex functions can be deduced from the pointwise convergence of their regularizations precisely when the $w^*$ and the norm topologies agree on the dual sphere. This extends some known results and answers an open question.

INTRODUCTION

Let $(X, ||·||)$ be a real Banach space and let $\Gamma(X)$ denote the proper convex lower semicontinuous functions on $X$. Throughout this paper, we will assume all functions are in $\Gamma(X)$ without always explicitly mentioning it. The following very useful and now classical notion of epi-convergence for sequences of functions was introduced by Mosco in [M1]. A sequence $\{f_n\}_{n=1}^\infty \subset \Gamma(X)$ is said to converge Mosco to $f \in \Gamma(X)$ if the following two conditions are satisfied.

(M1) For every $x \in X$, $\limsup_{n \to \infty} f_n(x_n) \leq f^*(x)$ for some sequence $x_n \to x$.

(M2) For every $x \in X$, $\liminf_{n \to \infty} f_n(x_n) \geq f(x)$ for every sequence $x_n \rightharpoonup x$.

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† NSERC postdoctoral fellow.
One of the important properties of Mosco convergence is that, unlike pointwise convergence, it is preserved under Fenchel conjugation in reflexive spaces [M2]. This bicontinuity theorem is one of the main reasons for the huge success of Mosco convergence in questions of optimization and differentiability (see [At]); for some recent applications to second order differentiability of convex functions see [BN, Sect. 6].

It was shown in [BB] that Mosco convergence has serious defects outside of reflexive spaces. Fortunately, the following notion of set convergence—which agrees with Mosco convergence in reflexive spaces—overcomes most of the shortcomings of Mosco convergence in nonreflexive spaces. A net of closed convex sets \( \{ C_s \}_s \) is said to converge slice to the closed convex set \( C \) if for every bounded convex set \( W \), \( d(W, C_s) \to d(W, C) \), where \( d(A, B) = \inf \{ \|a - b\|: a \in A, b \in B \} \) for any two sets \( A, B \). Slice convergence is invariant under renorming (see [Be4, Theorem 3.1]) and so one can say a net \( \{ f_s \}_s \subset \Gamma(X) \) converges slice to \( f \in \Gamma(X) \) if their epigraphs converge slice in \( X \times \mathbb{R} \) without specifying which equivalent norm is used on \( X \times \mathbb{R} \). Some of the notable features of slice convergence in general spaces include its analog of Mosco's [M2] important bicontinuity theorem and its applications to the convergence of subdifferentials; see [Be3, AB].

Even for sequences of Lipschitz convex functions, pointwise convergence fails to imply slice convergence or even weaker notions of epi-convergence; see [SW'] and Example 3.3(a). Because of this and the fact that for large parameters \( \mu \) the Lipschitz regularization of a Lipschitz function is itself, one might guess that no information about slice convergence of a sequence of functions can be obtained by studying pointwise convergence of their regularizations (where for \( \mu > 0, f \in \Gamma(X) \) and \( \| \cdot \| \) the norm on \( X \), we define and denote the Lipschitz regularization with parameter \( \mu \) as \( f(x, \mu) = \inf \{ f(z) + \mu \| x - z \|: z \in X \} \)). This notwithstanding, Azé [Az] and more recently Beer [Be5] have shown in certain circumstances a sequence and respectively a net of functions converges slice provided their regularizations converge pointwise. In [Az, Theorem 2.2], the space is reflexive and a good norm is used while in [Be5, Theorem 3.3] pointwise convergence of the regularizations of both the functions and their conjugates is needed. However, several things have remained open. For instance, what information does pointwise convergence of the Lipschitz regularizations alone provide (i.e., without any assumptions on the norm or on the regularizations of the conjugates)? It was also unknown if the result of [Az] would be valid in nonreflexive spaces, or if the conditions on the norm could be weakened. The goal of this paper is address these questions.

Our central result (Theorem 4.1) is that given a sequence of convex functions on a separable Banach space, pointwise convergence of their Lipschitz regularizations can be expressed in terms of variants of two
conditions used by Attouch and Beer in [AB, Theorem 3.1] to characterize slice convergence. The keys to our nonseparable result (Theorem 4.3) are a separable reduction argument and the observation (Corollary 3.2) that in separable spaces the Attouch–Beer-type conditions are also equivalent to the following notion of set convergence introduced by Wijsman in [W]. We will stay a net of closed convex sets \( \{ C_x \} \) converges Wijsman to the closed convex set \( C \) if \( d(x, C_x) \to d(x, C) \) for all \( x \in X \). For Wijsman convergence it is crucial to specify which norm is being used; see [Be1, Be4, BFi]. Thus we will study when the epigraphs of a net of functions in \( \Gamma(X) \) converge Wijsman to an epigraph of a function in \( \Gamma(X) \) with respect to a fixed norm on \( X \times \mathbb{R} \). However, our characterization of Wijsman convergence in terms of the Attouch–Beer-type conditions shows that there is a certain amount of flexibility in which norm is chosen on \( X \times \mathbb{R} \). It had been previously unclear, at least to us, whether Wijsman convergence with respect to an \( l_p \)-product of the norms on \( X \) and \( \mathbb{R} \) implies Wijsman convergence with respect to every \( l_q \)-product of the norms for \( 1 \leq q \leq \infty \) (Theorem 4.3 and Example 3.3(a) show that it does for \( 1 < p \leq \infty \) but not for \( p = 1 \)).

There are some nice known connections between Wijsman convergence and renorming. One such result is that a net of convex sets converges Wijsman with respect to every equivalent renorm if and only if it converges slice [Be4, Theorem 3.1]. In a different direction, one can ask: If a norm is nice enough, does Wijsman convergence with respect to this norm imply slice convergence? Such a result for reflexive spaces was obtained in [BFi]; see also [Be2]. Recently, the result of [BFi] was extended to nonreflexive spaces. First, a dual norm is said to be \( w^* \)-Kadec \((w^*+\tau\text{-Kadec})\) if the \( w^* \) and norm \((w^* \text{ and Mackey})\) topologies agree on its sphere. In [BV, Theorems 2.1 and 2.3], it is shown that Wijsman convergence implies slice (Mosco) convergence for all nets of convex sets in \( X \) precisely when the dual norm on \( X^* \) is \( w^* \)-Kadec \((w^*+\tau\text{-Kadec})\). This with our results on Lipschitz regularizations yields the following noteworthy corollary. If the dual norm is \( w^* \)-Kadec, then a net of functions converges slice provided their regularizations converge pointwise (Corollary 4.5). Typical examples of spaces whose duals admit \( w^* \)-Kadec norms are weakly compactly generated Asplund spaces (see [DGZ, Chapter VII]).

Our notation is quite standard. For a space \((X, \| \cdot \|)\), we let \( B_x = \{ x : \| x \| \leq 1 \} \), \( B_r = \{ x : \| x \| \leq r \} \), and \( S_x = \{ x : \| x \| = 1 \} \). The subdifferential of a convex function \( f \) at \( x_0 \) is defined by \( \partial f(x_0) = \{ A \in X^* : f(x) \geq f(x_0) + A(x - x_0) \text{ for all } x \in X \} \); and the \( \varepsilon \)-subdifferential is \( \partial_\varepsilon f(x_0) = \{ A \in X^* : f(x) \geq f(x_0) + A(x - x_0) - \varepsilon \text{ for all } x \in X \} \). The conjugate of a function \( f \in \Gamma(X) \) is denoted by \( f^* \) where \( f^*(y) = \sup_x \{ y(x) - f^*(x) : x \in X \} \). The indicator function of a set \( C \) is denoted by \( \delta_C \), where \( \delta_C(x) = 0 \text{ if } x \in C \) and \( +\infty \) otherwise.
1. Preliminary Lemmas

This section contains some elementary lemmas which we will need later. Although many of these lemmas are probably known (see [Be5, FP, HU1, HU2, La]), they are scattered throughout the literature and are not always stated in forms useful for our development. The first lemma, which is contained in [FP, Proposition 1.1], will be crucial in many of our proofs.

**Lemma 1.1.** The following are equivalent:

(a) \( \mu B_{x^*} \cap \partial f(x_0) \neq \emptyset \);
(b) \( f(x_0) = f(x_0, \mu) \);
(c) \( f(x_0, \mu) \) is finite and \( \partial f(x_0, \mu) = \partial f(x_0) \cap \mu B_{x^*} \).

**Proof.** (a) \( \Rightarrow \) (b): Suppose \( y_0 \in \partial f(x_0) \) and \( \| y_0 \| \leq \mu \). Now
\[
 f(x_0) \leq f(x) + y_0(x_0 - x) \leq f(x) + \mu \| x_0 - x \| \quad \text{for all} \quad x \in X.
\]
Thus \( f(x_0) = \inf \{ f(x) + \mu \| x_0 - x \| : x \in X \} = f(x_0, \mu) \).

(b) \( \Rightarrow \) (c): If \( y_0 \in \partial f(x_0, \mu) \), then
\[
y_0(x - x_0) \leq f(x, \mu) - f(x_0, \mu) \leq f(x) - f(x_0, \mu) = f(x) - f(x_0).
\]
Since \( f \) is \( \mu \)-Lipschitz [HU2, Corollary 1], this shows \( y_0 \in \partial f(x_0) \cap \mu B_{x^*} \).

For the reverse inclusion suppose \( y_0 \in \partial f(x_0) \cap \mu B_{x^*} \). Now, for any \( x \in X \),
\[
f(x, \mu) = \inf \{ f(v) + \mu \| v - x \| : v \in X \} \\
\geq \inf \{ f(x_0, \mu) + y_0(v - x_0) + \mu \| v - x \| : v \in X \} \\
= f(x_0, \mu) + y_0(x - x_0).
\]
Thus \( y_0 \in \partial f(x_0, \mu) \).

(c) \( \Rightarrow \) (a): This follows because \( f(\cdot, \mu) \) is \( \mu \)-Lipschitz.

**Lemma 1.2.** If \( y_0 \in \text{range}(\partial f) \), \( \mu = \| y_0 \| \) and \( f_\alpha(\cdot, \mu) \) converges pointwise to \( f(\cdot, \mu) \), that is, \( f_\alpha(x, \mu) \to f(x, \mu) \) for each \( x \in X \), then there is an \( x_0 \) such that \( f_\alpha(x) \geq -a - \mu \| x \| \) for some \( a \geq 0 \) and all \( \alpha \geq x_0 \).

**Proof.** Let \( y_0 \in \partial f(x_0) \), then \( f(x_0, \mu) \) is finite valued by Lemma 1.1. By the hypothesis, we have \( f_\alpha(x_0, \mu) \to f(x_0, \mu) \). Now \( f_\alpha(x_0, \mu) \geq f(x_0, \mu) - 1 \) for all \( \alpha \geq x_0 \). Thus
\[
f_\alpha(x) + \mu \| x_0 - x \| \geq f(x_0, \mu) - 1 \quad \text{for all} \quad \alpha \geq x_0.
\]
Hence \( f_\alpha(x) \geq a - \mu \| x \| \) for all \( \alpha \geq x_0 \), where \( a = |f(x_0, \mu) - 1| + \mu \| x_0 \| \).
The next two lemmas provide information about the number $\mu^* = d(0, \text{dom } f^*)$.

**Lemma 1.3.** If $\mu^* = d(0, \text{dim } f^*)$, then

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = -\mu^*.$$  

**Proof.** Let us assume by the way of contradiction that

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} > -\mu^*.$$  

Since functions in $\Gamma(X)$ are bounded below on bounded sets, this shows there exist $\alpha < \mu^*$ and $\alpha > 0$ such that

$$f(x) \geq -\alpha - \alpha \|x\| \quad \text{for all } x \in X.$$  

By the sandwich theorem (see [Bor] and the references therein) there exist $A \in X^*$ and $t \in \mathbb{R}$ such that

$$-\alpha - \alpha \|x\| \leq A(x) + t \leq f(x) \quad \text{for all } x \in X.$$  

It follows that $\|A\| \leq \alpha$ and $A \in \text{dom } f^*$; this is a contradiction since $\alpha < \mu^*$. Thus $\liminf_{\|x\| \to \infty} f(x) / \|x\| \leq -\mu^*$. On the other hand, given $y \in \text{dom } f^*$ with $\|y\| = \mu$, it follows that

$$f(x) \geq y(x) - f^*(y) \geq -\mu \|x\| - f^*(y).$$  

Hence $\liminf_{\|x\| \to \infty} f(x) / \|x\| \geq -\mu$ for all $\mu > -\mu^*$.

**Lemma 1.4.** If $\mu^* = d(0, \text{dom } f^*)$, then for $\lambda > \mu^*$, there exists $y \in \text{range}(\partial f)$ with $\|y\| \leq \lambda$. In particular, if $f_\alpha(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > \mu^*$ there exists an $\alpha > 0$ and $x_0$ such that $f_\alpha(x) \geq -\alpha - \lambda \|x\|$ for all $x \in X$ and all $\lambda \geq x_0$.

**Proof.** Let $\phi \in \text{dom } f^*$ with $\|\phi\| < \lambda$. Choose $\varepsilon > 0$ such that $\|\phi\| + \sqrt{\varepsilon} \leq \lambda$. It follows from the definition of $f^*$ that there is an $x_0 \in X$ with $\phi \in \partial f(x_0)$. By the Brøndsted–Rockafellar theorem [Ph, Theorem 3.18] there is a $y \in \text{range}(\partial f)$ with $\|y - \phi\| \leq \sqrt{\varepsilon}$. Combining this with Lemma 1.2 provides the "in particular" statement.

With Lemma 1.4 at our disposal, we are ready to prove the following important fact.
Proposition 1.5. Suppose for each \( \mu_k \) in some sequence \( \mu_k \uparrow \infty \), the net of functions \( f_\mu(\cdot, \mu_k) \) converges pointwise to \( f(\cdot, \mu) \). If \((x, r) \in \text{epi} f\), then \( d((x, r), \text{epi} f) \to 0 \) for any equivalent norm on \( X \times \mathbb{R} \); for sequences of functions this says precisely that \((M_i)\) holds.

Proof. Fix \((x_0, r) \in \text{epi} f\) and assume \( \mu > d(0, \text{dom} f^*) \). According to Lemma 1.4, there is an \( z_0 \) such that \( f_\mu(x) \geq -a - \mu \|x\| \) for some \( a \geq 0 \) and all \( x \geq z_0 \). Choose \( x_k \geq z_0 \) such that

\[
f_\mu(x_0, \mu_k) < f(x_0, \mu_k) + \frac{1}{k} \quad \text{for} \quad x \geq x_k.
\]

Let \( x_{k, k} \) satisfy \( f_\mu(x_{k, k}) + \mu_k \|x - x_{k, k}\| < f(x_0, \mu_k) + 1/k \leq f(x_0) + 1/k \) for \( x \geq x_k \). Now

\[-a - \mu_1 \|x_{k, k}\| + \mu_k \|x_0 - x_{k, k}\| \leq f(x_0) + \frac{1}{k}.
\]

From this, given any \( \varepsilon > 0 \), one can find \( k \) such that \( \limsup_k \|x_0 - x_{k, k}\| \leq \varepsilon \), since \( \mu_k \to \infty \). Moreover, \( f_\mu(x_{k, k}) \leq f(x_0) + 1/k \leq r + 1/k \) for \( x \geq x_k \) and so the proposition follows.

The following lemma will be useful in cases when we have information about \( f(x, \lambda) \) for \( \lambda > \mu \) (or \( \lambda < \mu \)).

Lemma 1.6.

(a) \( f(x, \mu) = \sup_{\lambda < \mu} f(x, \lambda) \) for all \( \mu > \mu^* \).

(b) \( f(x, \mu) = \inf_{\lambda > \mu} f(x, \lambda) \) for all \( \mu \geq \mu^* \).

Proof. (a) Fix \( \mu^* < \lambda_0 < \mu \), let \( \lambda_n > \lambda_0 \) and \( \lambda_n \uparrow \mu \). Choose \( z_n \) such that

\[
f(x, \lambda_n) \geq f(z_n) + \lambda_n \|x - z_n\| - \frac{1}{n}.
\]

According to Lemma 1.4, \( f(u) \geq -a - \lambda_0 \|u\| \) for some \( a \geq 0 \) and all \( u \in X \). Thus it follows that \( \{\|x - z_n\|\}_{n=1}^\infty \) is bounded and so \( (\mu - \lambda_n)\|x - z_n\| \to 0 \) which means

\[
f(x, \mu) \leq \lim_{n \to \infty} \inf \{f(z_n) + \lambda_n \|x - z_n\| + (\mu - \lambda_n)\|x - z_n\|\}
\]

\[
\leq \lim_{n \to \infty} \inf f(x, \lambda_n) \leq f(x, \mu).
\]
(b) Choose \( z_n \) such that \( f(x, \mu) = \lim_{n \to \infty} f(z_n) + \mu \| x - z_n \| \). Now choose \( \lambda_n \) satisfying \( \lambda_n > \mu \) satisfying \( \lambda_n \| x - z_n \| < 1/n \), then

\[
\begin{align*}
f(x, \mu) &= \lim_{n \to \infty} (f(z_n) + \lambda_n \| x - z_n \|) \geq \liminf_{n \to \infty} f(x, \lambda_n) \geq f(x, \mu).
\end{align*}
\]

2. Convergence of Regularizations and Slice Convergence: Special Cases

In this section we show that slice convergence of a sequence of functions can be deduced from the pointwise convergence of their regularizations in two special but new and rather general cases. Completely general results will be given in the fourth section; nevertheless, we have chosen to include these special cases because their proofs are much simpler than that of the general case and they illustrate many of the key techniques. First we need to introduce the following two conditions.

\((\text{AB}_1)\) If \( x_0 \in \text{dom}(\partial f) \), there exist \( x_n \to x_0 \) such that \( \limsup_{n \to \infty} f_n(x_n) \leq f(x_0) \).

\((\text{AB}_2)\) If \( y_0 \in \text{range}(\partial f) \), there exist \( y_n \in X^* \), with \( \| y_n - y_0 \| \to 0 \) and \( \limsup_{n \to \infty} f_n^*(y_n) \leq f^*(y_0) \).

Observe that these conditions are independent of the particular norm used on \( X \). The importance of these conditions stems from the following theorem of Attouch and Beer.

**Theorem 2.1.** [\text{AB}, Theorem 3.1] Let \( X \) be a Banach space and suppose \( f_n, f \in \Gamma(X) \). Then \( f_n \) converges slice to \( f \) if and only if \( (\text{AB}_1) \) and \( (\text{AB}_2) \) are satisfied.

Recall that a norm \( \| \cdot \| \) is locally uniformly round (LUR) if \( \| x_n - x \| \to 0 \) whenever \( \| x_n \| \to \| x \| = 1 \) and \( \| x_n + x \| \to 2 \).

**Theorem 2.2.** Suppose \( f_n, f \in \Gamma(X) \) and the dual norm on \( X^* \) is LUR. If \( f_n(\cdot, \mu) \) converges pointwise to \( f(\cdot, \mu) \) for all \( \mu \geq d(0, \text{dom } f^*) \), then \( f_n \) converges slice to \( f \).

**Proof.** Fix \( y_0 \in \partial f(x_0) \) and let \( \mu = \| y_0 \| \). Now choose \( x_n \) such that \( y_0(x_n - x_0) > \mu(1 - 1/n) \) and \( \| x_n - x_0 \| = 1 \). According to Lemma 1.1, \( y_0 \in \partial f(x_0, \mu) \); thus

\[
\mu \left( 1 - \frac{1}{n} \right) < y_0(x_n - x_0) \leq f(x_n, \mu) - f(x_0, \mu) \leq \mu \| x_n - x_0 \| \leq \mu.
\]

(2.1)
By (2.1) and pointwise convergence, choose $k_n$ such that
\[
f_k(x_n, \mu) - f_k(x_n, \mu) \leq -\mu \left(1 - \frac{1}{n}\right) \quad \text{for} \quad k \geq k_n. \tag{2.2}
\]

Assume $k_1 < k_2 < \cdots$ and let $y_k \in \partial f_k(x_n, \mu)$ for $k_n \leq k < k_{n+1}$. Then
\[
y_k(x_0 - x_n) \leq f_k(x_0, \mu) - f_k(x_n, \mu) \quad \text{for} \quad k_n \leq k < k_{n+1}. \tag{2.3}
\]

According to (2.1)-(2.3):
\[
(y_k + y_0)(x_n - x_0) \geq 2\mu \left(1 - \frac{1}{n}\right) \quad \text{for} \quad k_n \leq k < k_{n+1}. \tag{2.4}
\]

Because $\|y_k\| \leq \mu$ and the dual norm is LUR, we have $y_k \to y_0$. Moreover, since $y_k \in \partial f_k(x_n, \mu)$ for $k_n \leq k < k_{n+1}$, we obtain
\[
f_k^*(y_k) = \sup_{z} (y_k(z) - f_k(z))
\leq \sup_{z} (y_k(z) - f_k(z, \mu)) \quad [\text{since } f_k(z, \mu) \leq f_k(z)]
= (f_k(z, \mu))^*(y_k)
y_k(x_n) - f_k(x_n, \mu)
= \{y_0(x_0) - f(x_0, \mu)\} + \{y_k - y_0\}(x_n)
+ \{f(x_0, \mu) - f_k(x_n, \mu)\} + \{y_0(x_n - x_0)\}. \tag{2.5}
\]

Now observe:
\[
f^*(y_0) = y_0(x_0) - f(x_0) = y_0(x_0) - f(x_0, \mu). \tag{2.6}
\]

Because $\{x_n\}_{n=1}^{\infty}$ is bounded and $\|y_k - y_0\| \to 0$, we have
\[
(y_k - y_0)(x_n) \to 0. \tag{2.7}
\]

Since $f_k(x_0, \mu) \to f(x_0, \mu)$, (2.2) implies
\[
f(x_0, \mu) - f_k(x_n, \mu) \to -\mu. \tag{2.8}
\]

Using (2.6)-(2.8) and the fact that $y_0(x_n - x_0) \to 0$ in (2.5), one obtains
\[
\limsup_{k \to \infty} f_k^*(y_k) \leq f^*(y_0).
\]

Thus (AB$_2$) holds and (AB$_1$) follows from Proposition 1.5. Applying [AB, Theorem 3.1] establishes the theorem. \[\square\]
**Lemma 2.3.** Suppose \( f \in \Gamma(X) \) and \(-\infty < f(x_0, \mu) < f(x_0) \). If \( y_0 \in \partial f(x_0, \mu) \), then \( \| y_0 \| = \mu \).

**Proof.** Fix \( r \) with \( f(x_0, \mu) < r < f(x_0) \). Since \( f \) is lower semicontinuous, there is a \( \delta > 0 \) such that \( f(x) > r \) whenever \( \| x - x_0 \| < \delta \). Let \( \varepsilon > 0 \), satisfy \( f(x_0, \mu) + \delta \varepsilon < r \) and choose \( v \in X \) such that

\[
f(v) + \mu \| x_0 - v \| \leq f(x_0, \mu) + \delta \varepsilon.
\]

Now, \( \| v - x_0 \| \leq \delta \) and

\[
f(v, \mu) - f(x_0, \mu) \leq f(v) - f(x_0, \mu) \leq -\mu \| v - x_0 \| + \delta \varepsilon.
\]

From this, for \( y \in \partial f(x_0, \mu) \), we have \( y(x_0 - v) \geq \mu \| v - x_0 \| - \varepsilon \). Since \( f(\cdot, \mu) \) is \( \mu \)-Lipschitz we conclude \( \| y \| = \mu \). \( \square \)

The following theorem improves [Az, Theorem 2.2] where the \( X \) is assumed to be reflexive and the norms on \( X \) and \( X^* \) are both strictly convex and weak Kadec.

**Theorem 2.4.** Suppose \( f_n, f \in \Gamma(X) \) and the dual norm on \( X^* \) is \( w^* \)-Kadec. If \( f_n(\cdot, \mu) \) converges pointwise to \( f(\cdot, \mu) \) for all \( \mu \geq \mu^* = d(0, \text{dom } f^*) \), then \( f_n \) converges Mosco to \( f \).

**Proof.** According to Proposition 1.5, (M1) holds, so we verify (M2). The details for this were obtained from a careful analysis of [Az]. Fix \( x \in X \) and suppose \( x_n \rightharpoonup x \), we will show \( \liminf_{n \to \infty} f_n(x_n) \geq f(x) \). By passing to a subsequence, we may and do assume \( \lim_{n \to \infty} f_n(x_n) = \liminf_{n \to \infty} f_n(x_n) \).

Suppose there exists \( \mu \geq \mu^* \) such that \( f(x, \mu) < f(x) \) and let \( y_n \in \partial f_n(x, \mu) \). Now there is a subsequence \( \{ y_k \}_k \) such that \( y_k \rightharpoonup y \) for some \( y \), because any dual space with a dual \( w^* \)-Kadec norm has \( w^* \)-sequentially compact dual ball; e.g., [BV, Remark 1.5(a)]. Since \( f_n(\cdot, \mu) \) converges pointwise to \( f(\cdot, \mu) \), it follows that \( y \in \partial f(x, \mu) \). Because \( f(x, \mu) < f(x) \), Lemma 2.3 implies \( \| y \| = \mu \). Thus by the Kadec property of the dual norm, \( y_k \rightharpoonup y \). Since \( y_k \in \partial f_k(x, \mu) \), we have

\[
f_k(x_k) \geq f_k(x_k, \mu) \geq y_k(x_k - x) + f_k(x, \mu).
\]

Because \( x_k \rightharpoonup x \) and \( y_k \rightharpoonup y \), this shows that

\[
\liminf_{n \to \infty} f_n(x_n) = \lim_k f_k(x_k) = \lim_k f_k(x, \mu) = f(x, \mu) = f(x, \mu) = f(x).
\]

This yields \( \liminf_{n \to \infty} f_n(x_n) \geq \sup \{ f(x, \mu) : f(x, \mu) < f(x) \} = f(x) \). Thus (M2) holds provided \( f(x, \mu) < f(x) \) for some \( \mu > \mu^* \).
In the case $f(x, \mu) = f(x)$ for all $\mu > \mu^*$, then $f(x, \mu^*) = f(x)$ by Lemma 1.6(b). Since $\mu^* = \inf\{ \| y \| : y \in \text{range}(\partial f) \}$, one has for $y \in \partial f(x, \mu^*)$ that $\| y \| = \mu^*$ (this follows for instance from Lemma 1.3). Thus for $y_n \in \partial f_n(x, \mu^*)$, one can show, as in the previous paragraph, that $y_n \to y$ for some $y$, and some subsequence. Now (2.9) shows $\lim_{n \to \infty} f_n(x_n) \geq f(x, \mu^*) = f(x)$. Thus (M2) holds in this case as well. 

3. A CHARACTERISATION OF WIJSMAN CONVERGENCE IN SEPARABLE SPACES

In this section we will present a Wijsman convergence analog for separable spaces of [AB, Theorem 3.1] (see Theorem 2.1). For this result, we do not need any information concerning pointwise convergence of Lipschitz regularizations, so we are presenting it separately. Nevertheless, it will be crucial in developing our results for nonseparable spaces.

Let $(X, \| \cdot \|)$ be a Banach space, in order to obtain a Wijsman variant of the above theorem we need the following $w^*$-version of (AB2); recall that (AB1) and (AB2) are stated before Theorem 2.1. Note that this condition depends on the norm used on $X$.

(AB1*): If $y_0 \in \text{range}(\partial f)$, then there exist $y_n \in X^*$ with $y_n \rightharpoonup y_0$, $\| y_n \| \to \| y_0 \|$, and

$$\limsup_{n \to \infty} f_n^*(y_n) \leq f^*(y_0).$$

Let us say a norm $\| (\cdot, \cdot) \|$ on $X \times \mathbb{R}$ is compatible with $\| \cdot \|$ on $X$ if its dual satisfies:

$$\| (y_n, t) \| \to \| (y_0, t) \| \quad \text{whenever} \quad y_n \rightharpoonup y_0 \quad \text{and} \quad \| y_n \| \to \| y_0 \|. $$

For example, $\| (x, r) \| = \| (\| x \|, r) \|_p$ is compatible with $\| \cdot \|$ for every $1 \leq p \leq \infty$, where $\| (\cdot, \cdot) \|_p$ is the $l_p$-norm on $\mathbb{R}^2$.

**Theorem 3.1.** Let $(X, \| \cdot \|)$ be a Banach space, suppose $f_n, f \in \Gamma(X)$, and let $\| (\cdot, \cdot) \|$ be a norm on $X \times \mathbb{R}$.

(a) Suppose $\| (\cdot, \cdot) \|$ is compatible with $\| \cdot \|$. If (AB1) and (AB1*) are satisfied, then $\text{epi } f_n$ converges Wijsman to $\text{epi } f$ with respect to $\| (\cdot, \cdot) \|$.

(b) Suppose $X$ is separable and suppose the dual norm on $X^* \times \mathbb{R}$ satisfies:

$$(y_n, t_n) \rightharpoonup (y_0, -1) \quad \text{and} \quad \| (y_n, t_n) \| \to \| (y_0, -1) \| \quad \text{imply} \quad \| y_n \| \to \| y_0 \|. $$

If $\text{epi } f_n$ converges Wijsman to $\text{epi } f$, then (AB1) and (AB1*) hold.
Proof. (a) Let \((x, r) \in X \times \mathbb{R}\) be arbitrary and let \(C = \text{epi } f\), \(C_n = \text{epi } f_n\). It follows from [Ph, Theorem 3.17] that \((t, f(c)) : t \in \text{dom}(\partial f)\) is dense in the graph of \(f\). Consequently (AB\(_1\)) implies \(\limsup_{n \to \infty} d((x, r), C_n) \leq d((x, r), C)\). We now show \(\liminf_{n \to \infty} d((x, r), C_n) \leq d((x, r), C)\). Trivially we may assume \(d((x, r), C) > 0\). Now if \(0 < r < d((x, r), C)\), then \(d(F, C) > 0\) where \(F = (x, r) + B_r\). Thus by [Be3, Lemma 4.10], there is a \(y_0 \in \partial f(x_0)\) for some \(x_0 \in X\), such that \(F\) lies below the graph of \(f(x_0) + y_0 (\cdot - x_0)\); in other words,

\[
\sup_{C} A_0 + \rho \|A_0\| \leq A_0(x, r) \quad \text{where} \quad A_0 = (y_0, -1). \tag{3.1}
\]

By (AB\(_1^*\)), we can find \(y^*_n \underset{n}{\xrightarrow{*}} y_0\) such that \(\|y_n\| \to \|y_0\|\) and \(\limsup_{n \to \infty} f_n^*(y_n) \leq f^*(y_0)\). Because the norm on \(X \times \mathbb{R}\) is compatible with \(\|\cdot\|\), we have \(\|A_n\| \to \|A_0\|\) where \(A_n = (y_n, -1)\). For \(\varepsilon > 0\) given, fix \(n_0\) such that \(f_n^*(y_n) \leq f^*(y_0) + \varepsilon \|A_0\|\) and \(\|(A_n - A_0)(x, r)\| < \varepsilon \|A_0\|\) for \(n \geq n_0\). According to (3.1) for \(n \geq n_0\), we have

\[
\|A_n\| d((x, r), C_n) \geq A_n(x, r) - \sup_{C_n} A_n = A_n(x, r) - f_n^*(y_n)
\]

\[
\geq A_0(x, r) - f^*(y_0) - 2\varepsilon \|A_0\|
\]

\[
= A_0(x, r) - \sup_{C} A_0 - 2\varepsilon \|A_0\| \geq (\rho - 2\varepsilon) \|A_0\|.
\]

Since \(\|A_n\| \to \|A_0\|\), this shows \(\liminf_{n \to \infty} d((x, r), C_n) \geq \rho\) for any \(\rho < d((x, r), C)\) and completes the proof of (a).

(b) Let \(C_n = \text{epi } f_n\) and \(C = \text{epi } f\). Fix \((x_0, f(x_0)) \in C\), then by Wijsman convergence one has \((x_n, t_n) \in C_n\) such that \((x_n, t_n) \to (x_0, f(x_0))\). Now \(t_n \to f(x_0)\) and

\[
\limsup_{n \to \infty} f_n^*(y_n) \leq \lim_{t \to \infty} t_n = f(x_0),
\]

so (AB\(_1\)) holds. To establish (AB\(_1^*\)), we let \(y_0 \in \partial f(x_0)\). Now \((y_0, -1)\) attains its supremum on \(C\) at \((x_0, f(x_0))\). By [BV, Remark 2.2] (separability is used here), there exist \((y_n, t_n) \xrightarrow{w^*} (y_0, -1)\) with \(\|(y_n, t_n)\| = \|(y_0, -1)\|\) and

\[
\limsup_{n \to \infty} \sup_{C_n} \{y_n, t_n\} \leq \sup_{C} (y_0, -1).
\]

Now \(t_n \to -1\) so for \(\tilde{y}_n = y_n / |t_n|\), we have \((\tilde{y}_n, -1) \xrightarrow{w^*} (y_0, -1)\) and

\[
\limsup_{n \to \infty} \sup_{C_n} (\tilde{y}_n, -1) = \limsup_{n \to \infty} \left\{ \frac{1}{|t_n|} \sup_{C_n} (y_n, t_n) \right\} \leq \sup_{C} (y_0, -1).
\]
By the hypothesis on \( \| \cdot, \cdot \| \), we have \( \| y_n \| \to \| y_0 \| \). Finally,
\[
\limsup_{n \to \infty} f_n^*(\tilde{y}_n) = \limsup_{n \to \infty} \{ \sup_{x} \tilde{y}_n(x) - f(x) \} \\
= \limsup_{n \to \infty} \{ \sup_{x} (\tilde{y}_n, -1) \} \\
\leq \sup_{C} y_0, -1) = f^*(y_0).
\]
Thus (AB\(^*_n\)) holds.

The following corollary is an analog of [BV, Remark 2.2]. It has the advantage of showing that there is some flexibility regarding which product norm on \( X \times \mathbb{R} \) is used when checking Wijsman convergence of convex functions.

**Corollary 3.2.** Let \( (X, \| \cdot \|) \) be a separable Banach space and suppose \( f_n, f \in \Gamma(X) \). Then the following are equivalent.

(a) (AB\(^*_1\)) and (AB\(^*_f\)) are satisfied.

(b) epi \( f_n \) converges Wijsman to epi \( f \) with respect to every norm compatible with \( \| \cdot \| \).

(c) epi \( f_n \) converges Wijsman to epi \( f \) with respect to the norm \( \|(x, t)\| = \|(\|x\|, t)\|_p \) where \( 1 < p \leq \infty \) is fixed.

**Proof.** Theorem 3.1(a) shows (a) implies (b) while (b) trivially implies (c). One obtains (c) \( \Rightarrow \) (a) because the dual norm on \( X^* \times \mathbb{R} \) is of the form \( \|(f, t)\| = \|(\|f\|, t)\|_p \) with \( 1 \leq p < \infty \) which satisfies the condition of Theorem 3.1(b).

Building on ideas from [Be1], we provide examples which respectively show that one cannot let \( p = 1 \) in (c) of the above corollary and that (AB\(^*_1\)) and (AB\(^*_f\)) do not imply Wijsman convergence with respect to all norms on \( X \times \mathbb{R} \) which extend \( \| \cdot \| \).

**Example 3.3.** (a) Let \( \| \cdot \| \) be a norm on \( X \) whose dual is not sequentially \( w^*\)-Kadec ([BFa, Theorem 1]). Choose \( y_n, y \in S_{X^*} \) with \( y_n \rightharpoonup^* y \) but \( \inf_{n \in \mathbb{N}} \| y_n - y \| > 0 \). Now, there is an \( \varepsilon > 0 \), such that
\[
\sup_{n \in \mathbb{N}} \{ y_n(h) : \| h \| \leq 1 \text{ and } y(h) = 0 \} > 2\varepsilon \quad \text{for all } n.
\]

Set \( f_n(x) = d(x, C_n) \) and \( f(x) = d(x, C) \) where \( C_n = \{ x : y_n(x) = 1 \} \) and \( C = \{ x : y(x) = 1 \} \). It is easy to check that \( f_n \) converges pointwise to \( f \) (see [Be1, Theorem 4.3]). Now choose \( N \) such that \( 1/N < \varepsilon \). Then for
\[ \|x\| = \|x\| + N(y(x)), \] one has \( \|y\| < \varepsilon \) while (3.2) implies \( \|y_n\| > 2\varepsilon \). Thus \( d_x(0, C_n) < 1/\varepsilon \), but \( d_x(0, C_n) < 1/2\varepsilon \) for all \( n \). Let \( \|(x, t)\| \wedge \max\{\|x\|, |t|\} \), then \( d_x((0, -1/2\varepsilon), e_{\text{epi}} f_n) \leq 1/2\varepsilon \) for all \( n \), while \( d_x((0, -1/2\varepsilon), \text{epi} f) > 1/2\varepsilon \). Thus \( \text{epi} f_n \) does not converge Wijsman to \( \text{epi} f \) with respect to \( \|(\cdot, \cdot)\|_{\infty} \). However, \( \text{epi} f_n \) converges Wijsman to \( \text{epi} f \) in \( X \times \mathbb{R} \) with the norm \( \|(x, t)\|_1 = \|x\| + |t| \). To see this, let \( (x, r) \in X \times \mathbb{R} \).

If \( (x, r) \in \text{epi} f \), it is clear that \( d_x((x, r), \text{epi} f_n) \to 0 \) since \( f_n \) converges pointwise to \( f \). If \( d_x((x, r), \text{epi} f) = \rho > 0 \), using the fact that \( f \) is 1-Lipschitz on \( (X, \|\cdot\|) \), one can check that \( f(x) = \rho + r \). Thus \( f_n(x) \to \rho + r \). Now, since the \( f_n \) are also 1-Lipschitz on \( (X, \|\cdot\|) \), it follows that \( d_x((x, r), \text{epi} f_n) \to \rho \).

(b) Let \( (X, \|\cdot\|), y_n, y \) be as in (a). Now let \( f_n(x) = y_n(x) + 1 \) and \( f(x) = y(x) + 1 \). Since range(\( \partial f \)) = \{\( y \}\}, it is easy to see that \( (AB^+_\ast) \) is satisfied with \( y_n \in \partial f_n(x) \) for each \( x \in X \). The pointwise convergence of \( f_n \) to \( f \) shows that \( (AB^+_\ast) \) is satisfied. Define a norm \( \|(\cdot, \cdot)\|_B \) on \( X \times \mathbb{R} \) as the Minkowski functional of

\[ B = \{(x, t) : \|x\| \leq 1, |t| + |y(x)| \leq 1\} \]

Then \( \|(x, 0)\|_B = \|x\| \), but \( \text{epi} f_n \) does not converge Wijsman to \( \text{epi} f \) in the norm \( \|(\cdot, \cdot)\|_B \). To prove this, we first observe that \( d_B((0, 0), \text{epi} f) \equiv 1 \). Indeed, if \( f(x) = y(x) + 1 \) for \( t \leq 1 \) then \( y(x) \leq t - 1 \) which means \( \|y(x)\| \leq 1 - t \). Thus \( |t| + |y(x)| \geq t + 1 - t = 1 \) and so \( \|(x, t) - (0, 0)\|_B \geq 1 \). On the other hand, \( d_B((0, 0), \text{epi} f_n) \), it is true that \( \|h_n\| = \frac{1}{2} \) satisfying

\[ f_n(h_n) \leq 1 - \varepsilon \quad \text{and} \quad |y(h_n)| = 0. \]

So \( (h_n, 1 - \varepsilon) \in \text{epi} f_n \) and \( 1 - \varepsilon + |y(h_n)| = 1 - \varepsilon \). Therefore \( (1 - \varepsilon)^{-1}(h_n, 1 - \varepsilon) \in B \) which means \( \|f_n, 1 - \varepsilon - (0, 0)\|_B \leq 1 - \varepsilon \) for all \( n \).

4. Pointwise Convergence of Regularizations and Set Convergence

We begin with our key result. Nonseparable variants will be derived later using separable reduction arguments.

**Theorem 4.1.** Suppose \( f, f_n \in \Gamma(X) \) and suppose \( (X, \|\cdot\|) \) is a separable Banach space. Then the following are equivalent.

(a) \( f_n(\cdot, \mu) \) converges pointwise to \( f(\cdot, \mu) \) for all \( \mu > \mu^* = d(0, \text{dom} f^*) \).

(b) \( (AB^+_\ast) \) and \( (AB^+_\ast^\ast) \) are satisfied.

(c) \( \text{epi} f_n \) converges Wijsman to \( \text{epi} f \) with respect to every norm compatible with \( \|\cdot\| \).
Proof. Corollary 3.2 established the equivalence of (b) and (c); we now show (a) $\Rightarrow$ (b). By Proposition 1.5, we know that $(AB^*)$ is satisfied. Let us show $(AB^*)$ is satisfied. Let $y_0 \in \partial f(x_0)$ for some $x_0$. For all $\lambda \geq \|y_0\|$, Lemma 1.1 asserts $y_0 \in \partial f(x_0, \lambda)$ and $f(x_0, \lambda) = f(x_0)$. Let $\mu = \|y_0\|$ and let $\mu_n > \mu$ and $\mu_n \downarrow \mu$. Choose norm compact convex sets $K_n$, $n = 1, 2, 3, \ldots$ satisfying $x_0 \in K_1$, $K_1 \subset K_2 \subset K_3 \subset \cdots$ and $\bigcup_n K_n$ is norm dense in $X$. Since each $f_k (\cdot, \mu_n)$ is $\mu_n$-Lipschitz and $f_k (\cdot, \mu_n)$ converges pointwise to $f(\cdot, \mu_n)$, it follows that $f_k (\cdot, \mu_n)$ converges uniformly to $f(\cdot, \mu_n)$ for each $n$. Thus we choose positive integers $k_n$ such that for all $x \in K_n$ and $k \geq k_n$:

$$
\begin{align*}
 f(x, \mu_n) + \frac{1}{n} &\geq f_k (x, \mu_n) \geq f(x, \mu_n) - \frac{1}{2n} \\
 &\geq f(x_0, \mu_n) + y_0 (x - x_0) - \frac{1}{2n} \\
 &= f(x_0) + y_0 (x - x_0) - \frac{1}{2n}.
\end{align*}
$$

(4.1)

In particular, since $x_0 \in K_n$,

$$
f(x_0) + \frac{1}{n} \geq f_k (x_0, \mu_n) \geq f(x_0) - \frac{1}{n} \quad \text{for} \quad k \geq k_n.
$$

(4.1')

Using (4.1) with the sandwich theorem (see [Bor] and the references therein) we find functionals $y_k \in X^*$ and real numbers $r_k$ such that

$$
y_k (x) + r_k \leq f_k (x, \mu_n) \quad \text{for} \quad k_n \leq k < k_{n+1}, \text{ all } x \in X
$$

and

$$
y_k (x) + r_k \geq f(x_0) + y_0 (x - x_0) - \frac{1}{n} \quad \text{for all} \quad x \in K_n, k_n \leq k < k_{n+1}.
$$

(4.3)

In particular, because $K_n \subset K_m$ for $m \geq n$, we have

$$
y_k (x) + r_k \geq f(x_0) + y_0 (x - x_0) - \frac{1}{n} \quad \text{for all} \quad x \in K_n, k_n \leq k.
$$

(4.3')

Because $f_k (\cdot, \mu_n)$ is $\mu_n$-Lipschitz and $\{\mu_n\}_{n=1}^\infty$ is decreasing, (4.2) implies

$$
\|y_k\| \leq \mu_n \quad \text{for all} \quad k \geq k_n.
$$

(4.4)

We now show that $\liminf_{k \to \infty} y_k (h) \geq y_0 (h)$ for each $h \in X$. Indeed, fix $h \in X$ and let $\epsilon > 0$. Fix $n \in \mathbb{N}$ such that $d(K_n, x_0 + h) < \epsilon/4\mu_1$ and $4/n < \epsilon$. Let $k \geq k_n$ and $x_0 + h_1 \in K_n$ satisfy $\|h - h_1\| < \epsilon/4\mu_1$. According to (4.3') we have

$$
y_k (x_0 + h_1) + r_k \geq f(x_0) + y_0 (h_1) - \frac{1}{n} \quad \text{for} \quad k \geq k_n.
$$

(4.5)
On the other hand by (4.1') and (4.2), one has
\[
y_k(x_0) + r_k \leq f_k(x_0, \mu_n) \leq f(x_0) + \frac{1}{n} \quad \text{for} \quad k \geq k_n.
\] (4.6)\)

Subtracting (4.6) from (4.5) yields
\[
y_k(h_1) \geq y_0(h_1) - \frac{2}{n} \quad \text{for} \quad k \geq k_n.
\] (4.7)

Thus for \(k \geq k_n\),
\[
y_k(h) \geq y_k(h_1) - \mu_n \|h - h_1\| \quad \text{[by (4.4)]}
\]
\[
\geq y_0(h_1) - \frac{2}{n} - \mu_n \|h - h_1\| \quad \text{[By (4.7)]}
\]
\[
\geq y_0(h) - \frac{2}{n} - 2\mu_n \|h - h_1\| \quad \text{[\(\|y_0\| < \mu_n\)]}
\]
\[
> y_0(h) - \varepsilon.
\]

Therefore \(\liminf_{k \to \infty} y_k(h) \geq y_0(h)\) for all \(h \in X\) as desired. By homogeneity, (i.e., \(y_k(-h) = -y_k(h)\) for \(k = 0, 1, 2, \ldots\) it directly follows that \(\limsup_{k \to \infty} y_k(h) \leq y_0(h)\) for all \(h \in X\). Thus \(y_k \rightharpoonup y_0\). By \(\text{w}^*\)-lower semicontinuity of the norm, we know \(\liminf_{k \to \infty} \|y_k\| \geq \|y_0\|\); on the other hand it was observed in (4.4) that \(\|y_k\| \leq \mu_n\) for \(k_n \leq k < k_{n+1}\). From this, \(\|y_k\| \to \|y_0\|\).

Finally, we need to show
\[
\limsup_{k \to \infty} f^*_k(y_k) \leq f^*(y_0).
\] (4.8)

For this, recall that for \(k_n \leq k < k_{n+1}\):
\[
y_k(x) + r_k \leq f_k(x, \mu_n); \quad \text{[by (4.2)]}
\]
\[
y_k(x_0) + r_k \geq f(x_0) - \frac{1}{n} \geq f_k(x_0, \mu_n) - \frac{2}{n}. \quad \text{[by (4.1'), (4.3)]}
\]

Hence \(y_k \in \partial_{2n} f_k(x_0, \mu_n)\) for \(k_n \leq k < k_{n+1}\). Consequently for \(k_n \leq k < k_{n+1}\) we have
\[
(f_k(\cdot, \mu_n))^*(y_k) \leq y_k(x_0) - f_k(x_0, \mu_n) + \frac{2}{n}.
\]
Since $f_k(\cdot, \mu_n) \leq f_k(\cdot)$, it follows that
\[
\begin{align*}
&f_k^*(y_k) \leq (f_k(\cdot, \mu_n))^*(y_k) \leq y_k(x_0) - f_k(x_0, \mu_n) \\
&\quad + \frac{2}{n} \quad \text{for} \quad k_n \leq k < k_{n+1}.
\end{align*}
\]
Using this with (4.1') yields
\[
f_k^*(y_k) \leq y_k(x_0) - f(x_0) + \frac{3}{n} \quad \text{for} \quad k_n \leq k < k_{n+1}. \tag{4.9}
\]
Also, because $y_0 \in \partial f(x_0)$, one has
\[
f^*(y_0) = y_0(x_0) - f(x_0). \tag{4.10}
\]
By $w^*$-convergence, $y_k(x_0) \to y_0(x_0)$, thus (4.8) follows from (4.9) and (4.10). Therefore $(\text{AB}^+_\nu)$ is satisfied and so (a) implies (b).

(b) $\Rightarrow$ (a): We first show that $(\text{AB}^+_\nu)$ implies $\limsup_{\lambda \to \infty} f_\lambda(x, \mu) \geq f(x, \mu)$ for all $x \in X$ and $\mu \geq \mu^*$. For this, we need the following formula from [Be5, Lemma 2.2]:
\[
f(x, \mu) = \sup \{ y(x) - f^*(y) : \|y\| \leq \mu \} \quad \text{for} \quad \mu \geq \mu^*. \tag{4.11}
\]
Because there is a $y_0 \in \text{dom}(f^*)$ with $\|y_0\| < \mu$, (4.11) implies
\[
f(x, \mu) = \sup \{ y(x) - f^*(y) : \|y\| < \mu, y \in \text{dom}(f^*) \} \quad \text{for} \quad \mu \geq \mu^*. \tag{4.12}
\]
Combining (4.12) with [AB, Proposition 4.3], which says for $y \in \text{dom}(f^*)$ there is a sequence $\{y_n\}_{n=1}^\infty \subset \text{range}(\partial f)$ such that $y_n \to y$ and $f^*(y_n) \to f^*(y)$, we obtain
\[
f(x, \mu) = \sup \{ y(x) - f^*(y) : \|y\| < \mu, y \in \text{range}(\partial f) \} \quad \text{for} \quad \mu \geq \mu^*. \tag{4.13}
\]
Given any $x \in X$, $\varepsilon > 0$, and $\mu \geq \mu^*$, we use (4.13) to choose $y \in \text{range}(\partial f)$ such that $f(x, \mu) \leq y(x) - f^*(y) + \varepsilon$ and $\|y\| < \mu$. Since we are assuming $(\text{AB}^+_\nu)$, there are $n_0 \in \mathbb{N}$ and $y_n \in \partial f_n(x)$ such that $|f_n - y_n(x)| < \varepsilon$, $\|y_n\| < \mu$ and $f_n^*(y_n) < f^*(y) + \varepsilon$ for $n \geq n_0$. Thus for $n \geq n_0$, we obtain
\[
f(x, \mu) \leq y(x) - f^*(y) + \varepsilon \leq y_n(x) - f_n^*(y_n) + 3\varepsilon \leq f_n(x, \mu) + 3\varepsilon,
\]
where the last inequality follows from (4.11). Hence $\limsup_{\lambda \to \infty} f_\lambda(x, \mu) \geq f(x, \mu)$. 

Now we show that $\limsup_{n \to \infty} f_n(x, \mu)$ for all $x \in X$ and $\mu > \mu^*$. Let $x \in X$ and choose $\hat{x} \in X$ such that $f(\hat{x}) - \mu \| x - \hat{x} \| \leq f(x, \mu) + \varepsilon/3$. As a consequence of [Ph, Theorem 3.17], there is an $\hat{x} \in \text{dom}(\delta f)$ such that

$$\| \hat{x} - \hat{x} \| \leq \frac{\varepsilon}{3\mu} \quad \text{and} \quad |f(\hat{x}) - f(\hat{x})| \leq \frac{\varepsilon}{3}.$$ 

Then

$$f(\hat{x}) + \mu \| \hat{x} - x \| \leq f(x, \mu) + \varepsilon.$$ 

According to $(\text{AB}_1)$, we have $x_n \to \hat{x}$ with $\limsup_{n \to \infty} f_n(x_n) \leq f(\hat{x})$. From this,

$$\limsup_{n \to \infty} f_n(x_n) + \mu \| x_n - x \| \leq f(x, \mu) + \varepsilon,$$ 

which shows $\limsup_{n \to \infty} f_n(x, \mu) \leq f(x, \mu)$. 

The following lemma will enable us to obtain variants of Theorem 4.1 in nonseparable spaces.

**Lemma 4.2.** Suppose $\{f_n\}_{n=1}^\infty$ is a net in $\Gamma(X)$ and $f \in \Gamma(X)$. Let $X \times \mathbb{R}$ be endowed with a fixed norm. Suppose further $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > \mu^* = d(0, \text{dom} f^*)$. If $\text{epi} f_n$ does not converge Wijsman (slice) to $\text{epi} f$, then there is a separable $Z \subset X$ and a sequence $\{f_n\}_{n=1}^\infty$ such that $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > \mu^*$ (as regularizations on $Z$), but $\text{epi} f_n \cap (Z \times \mathbb{R})$ does not converge Wijsman (slice) to $\text{epi} f \cap (Z \times \mathbb{R})$ in the inherited norm.

**Proof.** First observe that $\liminf_{x} f_n(x) \geq f(x)$. Indeed, let $x_0 \in X$ be fixed and $r_0 < f(x_0)$. Pick $\mu_0$ with $f(x_0, \mu_0) > r_0 > r_0$ for some $r_1$. Fix $z_0$ with $f_n(x_0, \mu_0) > r_1$ for $z > r_0$. Now $f_n(x, \mu_0) > r_0$ for $\| x - x_0 \| < (r_1 - r_0)/\mu_0$. If $x_n \to x_0$, then $f_n(x_n) \geq f_n(x_0, \mu_0) > r_0$ eventually.

Let $C_x = \text{epi} f_n$ and $C = \text{epi} f$. From the observation in the previous paragraph one obtains $\limsup_x d((x, r), C_x) \leq d((x, r), C)$ for every $(x, r) \in X \times \mathbb{R}$. Since Wijsman convergence fails, it thus follows that there exists $(x, r) \in X \times \mathbb{R}$ such that $\liminf_x d((x, r), C_x) < d((x, r), C)$. By passing to a subnet we may assume that $d((x, r), C_x) + \delta < d((x, r), C)$ for all $x$ and some $\delta > 0$. Fix $\{x_n\}_{n=1}^\infty$ such that there exist $r_n \in \mathbb{R}$ with $(x_n, r_n) \in C_x$ and

$$d((x_n, r_n) - (x, r)) + \delta < d((x, r), C).$$ (4.14)

By Lemma 1.3, we can choose $\{z_n\}_{n=1}^\infty \in X$ such that

$$\liminf_{n \to \infty} \frac{f(z_n)}{\|z_n\|} = -\mu^*.$$
Thus if \( Z \) is any subspace containing \( \{ z_n \}_{n=1}^\infty \) and \( \overline{f} \) is the restriction of \( f \) to \( Z \), then \( d(0, \text{dom } \overline{f}^*) = \mu^* \).

Let \( \{ t_n \}_{n=1}^\infty \) be a set of numbers dense in \( (\mu^*, \infty) \). Let \( Z_1 \) be any separable space containing \( \{ z_1 \}_{n=1}^\infty \) and \( x \) where \( x \) is given by (4.14). We fix a countable dense set \( \{ z_{1,i} \}_{i=1}^\infty \) in \( Z_1 \) and let \( \alpha_i \) be an arbitrary index in the net. Suppose that \( Z_k \), \( \{ z_{k,i} \}_{i=1}^\infty \) and \( z_k \) have been chosen for \( k \leq n - 1 \). We choose \( x_n \geq x_{n-1} \) such that

\[
f(z_{i,j}, t_k) - \frac{1}{n} < f_n(z_{i,j}, t_k) < f(z_{i,j}, t_k) + \frac{1}{n} \quad \text{for } i, j, k \leq n - 1. \tag{4.15}
\]

Let \( \{ u_n,i \}_{i=1}^\infty \) be such that

\[
\inf_j \left\{ f(u_{n,i}) + t_k \| z_{i,j} - u_{n,i} \| \right\} = f(z_{i,j}, t_k) \quad \text{for } l \leq n - 1, i, k \in \mathbb{N}; \tag{4.16}
\]

\[
\inf_j \left\{ f_n(u_{n,i}) + t_k \| z_{i,j} - u_{n,i} \| \right\} = f_n(z_{i,j}, t_k) \quad \text{for } l, m \leq n - 1, i, k \in \mathbb{N}. \tag{4.17}
\]

Let \( Z_n = \text{span} \{ Z_{n-1} \cup \{ u_{n,i} \}_{i=1}^\infty \cup x_{x_n} \} \) where \( x_{x_n} \) is given by (4.14) and fix \( \{ z_n \}_{n=1}^\infty \) dense in \( Z_n \). We continue in this fashion, and we let the subspace \( Z \) be the norm closure of \( \bigcup Z_n \).

Let \( f_n \) and \( \overline{f} \in L(Z) \) denote the restrictions of \( f \) and \( \overline{f} \) to \( Z \). Similarly, let \( f_n(\cdot, \mu) \) and \( \overline{f}(\cdot, \mu) \) denote the regularizations of \( f_n \) and \( \overline{f} \) on \( Z \). It follows from (4.16) and (4.17) that

\[
\overline{f}(z_{i,j}, t_k) = f(z_{i,j}, t_k) \quad \text{and} \quad \overline{f}_n(z_{i,j}, t_k) = f_n(z_{i,j}, t_k) \quad \text{for all } i, j, k, n.
\]

Using this with (4.15), we obtain

\[
\lim_{n \to \infty} \overline{f}_n(z_{i,j}, t_k) = \overline{f}(z_{i,j}, t_k) \quad \text{for all } i, j, k. \tag{4.18}
\]

Since \( \{ z_{i,j} \}_{i,j}^\infty \) is norm dense in \( Z \), it follows from (4.18) and the Lipschitz property of regularizations that

\[
\lim_{n \to \infty} \overline{f}_n(z, t_k) = \overline{f}(z, t_k) \quad \text{for all } z \in N, k \in \mathbb{N}. \tag{4.19}
\]

Next, we use (4.19) to show that

\[
\lim_{n \to \infty} \overline{f}_n(z, \mu) = \overline{f}(z, \mu) \quad \text{for all } \mu > \mu^*, z \in Z. \tag{4.20}
\]
Fix $\mu > \mu^*$, $z \in Z$, and $\varepsilon > 0$. By Lemma 1.6(b), there is a $t_{k_0} > \mu$ such that $\tilde{f}(z, t_{k_0}) < \tilde{f}(z, \mu) + \varepsilon$. From (4.19) one obtains

$$\limsup_{n \to \infty} \tilde{f}_{s_n}(z, \mu) \leq \limsup_{n \to \infty} \tilde{f}_{s_n}(z, t_{k_0}) = \tilde{f}(z, t_{k_0}) < \tilde{f}(z, \mu) + \varepsilon$$

(4.21)

For the other inequality, according to Lemma 1.6(a) we can fix $t_{k_1} < \mu$ such that $\tilde{f}(z, t_{k_1}) > \tilde{f}(z, \mu) - \varepsilon$. Again by (4.19)

$$\liminf_{n \to \infty} \tilde{f}_{s_n}(z, \mu) \geq \liminf_{n \to \infty} \tilde{f}_{s_n}(z, t_{k_1}) = \tilde{f}(z, t_{k_1}) > \tilde{f}(z, \mu) - \varepsilon$$

(4.22)

Clearly (4.20) follows from (4.21) and (4.22). Finally, because $\{x_{s_n}^\ast\}_{n=1}^\infty \subseteq Z$ and $x \in Z$, it follows from (4.14) that $\tilde{f}$ does not converge Wijsman to $\tilde{f}$. The slice case can be proven similarly; one can also deduce it from the Wijsman case because if slice convergence fails there is a norm for which Wijsman convergence fails [Be4, Theorem 3.1].

For lack of a better term, we will say a norm $\|(\cdot, \cdot)\|$ on $X \times \mathbb{R}$ is separably compatible with $\|\cdot\|$ if for every separable subspace $Z \subset X$ the dual norm of restriction of $\|(\cdot, \cdot)\|$ to $Z \times \mathbb{R}$ satisfies

$$\|(z^\ast, t)\| \to \|(z^\ast, t)\|$$

whenever $z^\ast \xrightarrow{\mathbf{z^\ast}} z^\ast$ and $\|z^\ast\| \to \|z^\ast\|$.

It is not clear that compatible norms have this hereditary property. However, it is clear that $\|(x, r)\| = \|(\|x\|, r)\|_p$ is separably compatible with $\|\cdot\|$ for every $1 \leq p \leq \infty$. With this terminology, we are ready for an extension of Theorem 4.1 to nonseparable spaces. This theorem nicely complements [Be5, Theorems 3.3 and 3.7].

**Theorem 4.3.** Suppose $\{f_s\}_s$ is a net in $\Gamma(X)$ and $f \in \Gamma(X)$ for a Banach space $(X, \|\cdot\|)$. Then the following are equivalent.

(a) $f_s(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > \mu^* = d(0, \text{dom } f^*)$

(b) $\text{epi } f_s$ converges Wijsman to $\text{epi } f$ for every norm separably compatible with $\|\cdot\|$.

(c) $\text{epi } f_s$ converges Wijsman to $\text{epi } f$ with respect to the norm $\|(x, t)\| = \|(\|x\|, t)\|_p$ where $1 < p \leq \infty$ is fixed.

(d) $\text{epi } f_s(\cdot, \mu)$ converges Wijsman to $\text{epi } f(\cdot, \mu)$ with respect to every norm separably compatible with $\|\cdot\|$ for each $\mu > \mu^*$.

(e) For $\mu_k \uparrow \infty$, $\mu_k > \mu^*$, $\text{epi } f_s(\cdot, \mu_k)$ converges Wijsman to $\text{epi } f(\cdot, \mu_k)$ with respect to the norm $\|(x, t)\| = \|(\|x\|, t)\|_p$ where $1 < p \leq \infty$ is fixed.

**Proof.** (a) $\Rightarrow$ (b): Suppose $\text{epi } f_s$ does not converge Wijsman to $\text{epi } f$ with respect to some separably compatible norm $\|(\cdot, \cdot)\|$ on $X \times \mathbb{R}$. By Lemma 4.2,
there is a sequence $\tilde{f}_{n_k}$ and a separable subspace $Z$ such that $\text{epi} \tilde{f}_{n_k}$ does not converge Wijsman to $\text{epi} \tilde{f}$ in the inherited norm, while $\tilde{f}_{n_k}(\cdot, \mu)$ converges pointwise to $\tilde{f}(\cdot, \mu)$ for all $\mu > \mu^*$ (where as before, for a function $g, \tilde{g} = g|_Z$ and $\tilde{g}(\cdot, \mu)$ denotes the regularization of $\tilde{g}$ on $Z$). Since the inherited norm on $Z \times \mathbb{R}$ is compatible with the inherited norm on $Z$, this contradicts Theorem 4.1 and so (b) holds.

(b) $\Rightarrow$ (c): This is true because $\| (x, t) \| = \| (\| x \|, t) \|_\mu$ is separably compatible for every $1 \leq \mu \leq \infty$.

(c) $\Rightarrow$ (a): The outline of this proof is as follows. One can recursively construct a sequence $\{ x_n \}_{n=1}^\infty$ and a separable subspace $Z$ with the following properties. For $f_{n_k} \in \tilde{Z}$ and $\tilde{f}$ we have $d(0, \text{dom} \tilde{f}^*) = \mu^*$ but $f_{n_k}(\cdot, \mu)$ does not converge pointwise to $\tilde{f}(\cdot, \mu)$ for some $\mu > \mu^*$. This can be done in a similar fashion to Lemma 4.2. The argument of [BV, Theorem 1.2] shows that we can also ensure $\text{epi} \tilde{f}_{n_k}$ converges Wijsman to $\text{epi} \tilde{f}$ with respect to the restricted norm on $Z \times \mathbb{R}$. Since this norm is of the form $\| (x, t) \| = \| (\| x \|, t) \|_\mu$ for some $1 < \mu \leq \infty$, Corollary 3.2 and Theorem 4.1 together say that $\tilde{f}(\cdot, \mu)$ converges pointwise to $\tilde{f}(\cdot, \mu)$ for all $\mu > \mu^*$. This contradiction proves (a).

(a) $\Rightarrow$ (d): Fix $\lambda > \mu^*$ and let $g_\lambda(x) = f_\lambda(x, \lambda)$ and $g(x) = f(x, \lambda)$. Then $d(0, \text{dom} g^*) = \mu^*$ (this follows, for instance, from Lemma 1.3). Since $g$ is $\lambda$-Lipschitz for $\mu \geq \lambda$, $g_\lambda(x, \mu) = g_\lambda(x)$ and $g(x, \mu) = g(x)$. Since we are assuming (a), we conclude that $g_\lambda(\cdot, \mu)$ converges pointwise to $g(\cdot, \mu)$ for all $\mu \geq \lambda$. In the case $\mu^* < \mu < \lambda$, we have $g(x, \mu) = f(x, \mu)$ and $g_\lambda(x, \mu) = f_{\lambda}(x, \mu)$. Indeed,

$$g(x, \mu) = \inf_{v \in X} \{ g(v) + \mu \| x - v \| \} \leq \inf_{v \in X} \{ f(v) + \mu \| x - v \| \} = f(x, \mu) \leq \inf_{v \in X} \{ f(v) + \lambda \| u - v \| + \mu \| x - v \| \} = g(x, \mu).$$

Similarly, $g_\lambda(x, \mu) = f_{\lambda}(x, \mu)$. Thus by (a) the regularizations converge pointwise in this case as well. Because (a) implies (b) we deduce that $\text{epi} g_\lambda$ converges Wijsman to $\text{epi} g$ in every separably compatible norm. This shows that (d) holds.

Of course, (e) follows trivially from (d) so we show (e) $\Rightarrow$ (c). If $(x, r) \in \text{epi} f$, then $d((x, r), \text{epi} f) \to 0$ by Proposition 1.5. Hence for any $(x, r) \in X \times \mathbb{R}$,

$$\lim_{\lambda \to \infty} \sup_{\lambda \geq \mu} d((x, r), \text{epi} f) \leq d((x, r), \text{epi} f).$$

So suppose $d((x, r), \text{epi} f) = \rho > 0$. As in the proof of Theorem 3.1(a), there is a $y \in \partial f(x_0)$ for some $x_0$ such that $B_\rho + (x, r)$ and $\text{epi} f$ are separated by $(y, -1)$. For $\mu_k \geq \| y \|$, Lemma 1.1 says $y \in \partial f(x_0, \mu_k)$. Hence $(y, -1)$
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separates $\text{epi} f(\cdot, \mu_k)$ and $B_{\rho - c}$. From this, $d((x, r), \text{epi} f(\cdot, \mu_k)) \geq \rho - c$ and the Wijsman convergence of $\text{epi} f_n(\cdot, \mu_k)$ to $\text{epi} f(\cdot, \mu_k)$ implies

$$\lim_{x} \inf d((x, r), \text{epi} f_n(\cdot, \mu_k)) \geq \lim_{x} \inf d((x, r), \text{epi} f(\cdot, \mu_k)) \geq \rho - c.$$ 

Thus $\text{epi} f_n$ converges Wijsman to $\text{epi} f$.

In contrast to [Be5, Theorems 3.3 and 3.7] as well as (e) in the above theorem, we show that it does not suffice to have pointwise convergence of the regularizations for a sequence of $\mu_k$'s in Theorem 4.3(a), or even all parameters $\mu$ sufficiently large.

Remark 4.4. (a) Let $\| \cdot \|$ be a norm whose dual is not sequentially w*-Kadec. Choose $y_n, y \in S_{x^*}$ such that $y_n \rightharpoonup y$ but $\liminf_{n \to \infty} \| y_n - y \| > 0$. Set $f_n(x) = y_n(x) + 1$ and $f(x) = y(x) + 1$. Define $\| \cdot \|$ on $X$ by $\| x \| = \| x \| + \| y(x) \|$. It is easy to check directly that $\text{epi} f_n$ converges Wijsman to $\text{epi} f$ in $\| (x, t) \| = \max\{ \| x \|, |t| \}$ (alternatively, it is clear that $(AB_\gamma)$ and $(AB^*_\gamma)$ hold). However, $\text{epi} f_n$ does not converge Wijsman to $\text{epi} f$ in $\| (x, t) \| = \max\{ \| x \|, |t| \}$ (argue as in Example 3.3(b)). Since $\| \cdot \| \geq \| \cdot \|$, it follows that $f_n$ and $f$ are 1-Lipschitz on $(X, \| \cdot \|)$. Consider the regularizations

$$f_n(x, \mu) = \inf\{ f_n(x) + \mu \| x - z \| \} \quad \text{and} \quad f(x, \mu) = \inf\{ f(x) + \mu \| x - z \| \}.$$ 

For $\mu \geq 1$ it is clear that $f_n(x, \mu) = f_n(x)$ and $f(x, \mu) = f(x)$, consequently, $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$. From this one cannot use a sequence of $\mu_k$'s in Theorem 4.3(a). According to [Be5, Theorem 3.3], the regularizations of the conjugates cannot converge pointwise for large $\mu$. In fact, a simple check shows $f_n^* = 1_{\{y_n\}} - 1$ and $f^* = 1_{\{y\}} - 1$. Hence $f^*(y, \mu) = -1$ while $f_n^*(y, \mu) = \mu \| y_n - y \| - 1$ and so $f_n^*(y, \mu)$ does not converge to $f^*(y, \mu)$ for any $\mu > 0$.

(b) Part (a) shows that it is necessary to know that $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for small values of $\mu$ in Theorem 4.3(a); however, trivial examples show it is not sufficient. Indeed, set $f = 1_{\{0\}}$ and $f_n(x) = |x|$ for all $n$. Now $\mu = 0$ and for $0 \leq \mu \leq 1$, $f(x, \mu) = f_n(x, \mu) = \mu |x|$. However, it is clear that $\text{epi} f_n$ does not converge Wijsman to $\text{epi} f$ in any norm on $X \times \mathbb{R}$.

(c) According to Theorem 4.3, if $f_n$ and $f$ are $K$-Lipschitz for some $K > 0$, then $f_n$ converges pointwise to $f$ provided $\text{epi} f_n$ converges Wijsman to $\text{epi} f$ with respect to the norm $\| (x, t) \| = \max\{ \| x \|, |t| \}$ where $\| \cdot \|$ is an equivalent norm on $X$. However, part (a) above shows that the converse fails.

As a by-product of Theorem 4.3, we improve [Az, Theorem 2.2] and provide a complete answer to a question of Beer (see [Be5, Example 3.4 and discussion preceding it]).
COROLLARY 4.5. For a Banach space $X$, the following are equivalent.

(a) The dual norm on $X^*$ is $w^*\text{-Kadec}$ ($w^*\text{-Kadec}$).

(b) $f_n$ converges slice (Mosco) to $f$ whenever $\{f_n\}$ is a net in $\Gamma(X)$, $f \in \Gamma(X)$ and $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > \mu^*$.

Proof. (a) $\Rightarrow$ (b): Suppose $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > \mu^*$. By Theorem 4.3, epi $f_n$ converges Wijsman to epi $f$ in the norm $\| (x, t) \| = (\|x\|^2 + |t|^2)^{1/2}$. Since the dual of this norm is $w^*\text{-Kadec}$ ($w^*\text{-Kadec}$) it follows from [BV, Theorems 2.1 and 2.3] that the convergence is slice (Mosco). Note the slice can also be obtained independently of [BV, Theorem 2.1]. Indeed, if a norm has a $w^*\text{-Kadec}$ dual norm, it is easy to check that the restriction of this norm to any subspace has $w^*\text{-Kadec}$ dual norm (see, e.g., [BV, Proposition 1.4]). Now if the convergence were not slice, there would be a separable subspace and a sequence whose regularizations converge pointwise but the convergence would not be slice (Lemma 4.2). However, this is a contradiction with [AB, Theorem 3.1], because (AB1) and (AB2) must hold according to Theorem 4.1 and the fact that the dual of the restricted norm is $w^*\text{-Kadec}$.

(b) $\Rightarrow$ (a): This is known and is a net variant of Remark 4.4(a).

Let $f_n = \delta_{C_n}$ and $f = \delta_C$ where $C$, $C_n$ are closed convex sets. Now, $f_n(x, \mu) = \mu d(C, C_n)$ and $f(x, \mu) = \mu d(x, C)$. Thus $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > 0$ if and only if $C_n$ converges Wijsman to $C$. Consequently, the above corollary recaptures some results from [BV].

Using Mosco’s theorem ([M2]) we also obtain:

COROLLARY 4.6. For a reflexive Banach space $X$, the following are equivalent.

(a) For every sequence $\{f_n\}_{n=1}^\infty \subseteq \Gamma(X)$ and $f \in \Gamma(X)$, $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > d(0, \text{dom } f^*)$ if and only if $f_n^*(\cdot, \mu)$ converges pointwise to $f^*(\cdot, \mu)$ for all $\mu > d(0, \text{dom } f)$.

(b) The norms on $X$ and $X^*$ are weak Kadec.

Proof. (a) $\Rightarrow$ (b): If the norm on $X^*$ were not weak Kadec, it would not be sequentially weak Kadec. Thus the argument of Remark 4.4(a) would provide us with $f_n, f \in \Gamma(X)$ such that $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ for all $\mu > \mu^*$ such that $f_n^*(\cdot, \mu)$ does not converge to $f^*(\cdot, \mu)$ for any $\mu > 0$. We argue similarly if $X$ does not have a weak Kadec norm.

(b) $\Rightarrow$ (a): Because the norm on $X^*$ is weak Kadec, it follows from Corollary 4.5 that $f_n$ converges Mosco to $f$. By Mosco’s theorem [M2] we know that $f_n^*$ converges Mosco to $f^*$. This implies $f_n^*(\cdot, \mu)$ converges
pointwise to $f^*(\cdot, \mu)$ for all $\mu > d(0, \text{dom } f)$ [Az, Proposition 2.1]. Similarly $f_n(\cdot, \mu)$ converges pointwise to $f(\cdot, \mu)$ because the norm on $X$ is weak Kadec.

A further dual application of Theorem 4.3 is as follows. A net of $w^*$-closed convex sets $\{C_n\}_n$ in $X^*$ is said to converge dual slice to the $w^*$-closed set $C$ in $X^*$ if $d(W, C_n) \to d(W, C)$ for all $w^*$-compact convex sets $W \subset X^*$ where $d$ is measured with respect to a dual norm; see [Be3]. Arguing as in [BV, Theorem 2.1] one can prove:

**Theorem 4.7.** Suppose $X^*$ is separable and the norm on $X$ is weak Kadec. If a sequence of $w^*$-closed convex sets $\{C_n\}_{n=1}^\infty$ converges Wijsman to the $w^*$-closed convex set $C$, then $C_n$ converges dual slice to $C$.

This in combination with Theorem 4.3 provides us with the following result.

**Corollary 4.8.** Suppose $X^*$ is separable and the norm on $X$ is weak Kadec. If $f_n^*$ and $f^*$ are conjugate functions on $X^*$ such that $f_n^*(\cdot, \mu)$ converges pointwise to $f^*(\cdot, \mu)$ for all $\mu > \mu^*$, then $f_n^*$ converges dual slice to $f^*$.

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**References**


