The Occurrence of the Sequence 0123456789 within $\pi$.

In his primer on Intuitionism, Heyting [6] frequently relies on the occurrence or non-occurrence of the sequence 0123456789 in the decimal expansion of $\pi$ to highlight issues of classical versus intuitionistic (or constructivist) mathematics.1

At the time that Brouwer developed his theory (1908) and even at the time that [6] was written, it seemed well-nigh impossible that the first occurrence of any 10-digit sequence in $\pi$ could ever be determined.

The confluence of faster computers and better algorithms, both for $\pi$ and more importantly for arithmetic (fast-Fourier-transform-based, combined with Karatsuba, multiplication [3]), have rendered their intuition false. Thus, in June and July 1997, Yasumasa Kanada and Daisuke Takahashi at the University of Tokyo completed two computations of $\pi$ on a massively parallel Hitachi machine with $2^{10}$ processors. Details of Kanada and Takahashi's computation are lodged at www.cecm.sfu.ca/personal/jborwein/Kanada_50b.html. Included are some details of times, machinery used, digit distribution, etc. The key algorithms used are as in the recent survey in this journal [1], with the addition of significant numerical/arithmetic enhancements and subtle flow management.

During their computational tour-de-force, Kanada and Takahashi discovered the first occurrence of 0123456789 in $\pi$ beginning at the 17,387,594,880th digit (the '0') after the decimal point. It is worth noting that years become hours on such a parallel machine, and also that the labor of multiplying two seventeen-billion-digit integers together without FFT-based methods is to be measured in years. The underlying method reduces to roughly 300 such multiplications ([3],[2]). Add the likelihood of machines crashing during a many-year sequential computation. Hence without fast arithmetic and parallel computers, Brouwer and Heyting might indefinitely have remained safe in using this particular and somewhat natural example.

I may emphasize how out of reach the question appeared even 35 years ago with the following anecdote. Sometime after Shanks and Wrench computed 100,650 places of $\pi$ (in 1962 in 9 hours on an IBM 7090), Philip Davis asked Dan Shanks to fill in the blank in the sentence “mankind will never determine the $\cdots$ digit of $\pi$.” Shanks, apparently almost immediately, replied “the billionth.”

The Use of 0123456789 by Brouwer and Heyting

For completeness I list three of the “classical” examples:

1. The sequence $a = [2^{-n}]$ is a Cauchy sequence. Let the sequence $b = \{b_n\}$ be defined as follows: If the $n$th digit after the decimal point in the decimal expansion of $\pi$ is the 9 of the first sequence 0123456789 in this expansion, $b_n = 1$; in every other case $b_n = 2^{-n}$. $b$ differs from $a$ in at most one term, so $b$ is classically a Cauchy sequence.

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1To quote Carl Boyer, A History of Mathematics, John Wiley (1968) pp. 661–662: “According to Brouwer, language and logic are not presuppositions for mathematics, a subject that has its source in intuition that makes its concepts and inferences immediately clear to us; a statement that an object exists having a given property means that there is a known method that enables the object to be found or constructed in a finite number of steps. In particular, he argued the method of indirect proof, to which transfinite arithmetic had frequent recourse, is invalid. Ever since the time of Aristotle the three basic laws of logic had been held sacrosanct: (1) the law of identity, $A = A$; (2) the law of contradiction, $A$ cannot simultaneously be $B$ and not $B$; and (3) the law of the excluded middle (or tertium non datur). Brouwer denied the last of these laws of logic and refused to accept results based on it. For example, he asked the formalists whether it is true or false that the sequence of digits 123456789 (sic) occurs somewhere in the decimal expansion of $\pi$. Since no known method exists for making a decision, one cannot apply here the law of the excluded middle and assert that the proposition is either true or false.”
but as long as we do not know whether such a sequence 0123456789 occurs in \( \pi \), we are not able to find \( n \) such that \( b_{n+p} - b_n < 1/2 \) for every \( p \); we have no right to assert that \( \pi \) is a Cauchy sequence in our sense. ([6], page 16.)

2. A proof of the impossibility of the impossibility of a property is not in every case a proof of the property itself. It will be instructive to illustrate this by an example. I write the decimal expansion of \( \pi \) and under it the decimal fraction \( \rho = 0.333 \ldots \), which I break off as soon as a sequence of digits 0123456789 has appeared in \( \pi \). If the 9 of the first sequence 0123456789 in \( \pi \) is the 4th digit after the decimal point, \( \rho = (10^k - 1)/3 \cdot 10^k \). Now suppose that \( \rho \) could not be rational; then \( \rho = (10^k - 1)/3 \cdot 10^k \) would be impossible and no sequence could appear in \( \pi \); but then \( \rho = 1/3 \), which is also impossible. The assumption that \( \rho \) cannot be rational has led to a contradiction; yet we have no right to assert that \( \rho \) is rational, for this would mean that we could calculate integers \( p \) and \( q \) so that \( \rho = p/q \); this evidently requires that we can either indicate a sequence 0123456789 or demonstrate that no such sequence can occur. ([6], page 17, referring to [4].)

3. However, many other classical theorems are no longer valid. I state an example that a bounded monotone sequence need not be convergent. A simple counterexample is the sequence \( \{a_n\} \) which is defined as follows:

\[
a_n = 1 - 2^{-n}
\]

if among the first \( n \) digits in the decimal expansion of \( \pi \) no sequence 0123456789 occurs; \( a_n = 2 - 2^{-n} \) if among these \( n \) digits such a sequence does occur. Nobody knows if the limit of this sequence, if it exists, will be 1 or 2; so we are not allowed to say that this limit exists as a well defined real number generator. ([6], page 31.)

The sequence continues to occur, next at the 26,852,899,245th place. The first six occurrences begin at the 17,387,594,880; 26,852,899,245; 30,243,957,439; 34,549,153,953; 41,952,536,161; 43,280,964,000 place respectively. A simple calculation suggests that the probability of the first instance being within the first 25 billion places was over 90%, with the probability of it being in the first 7 billion under 50%.

Side notes: 00876543210 begins at the 42,321,758,803rd digit of \( \pi \); and the digits ending at 50 billion are

\[
85,133,987,127,510,930,042.
\]

At any rate, sometime in July 1997, the three examples above began to behave well. Needless-to-say, the strength of the intuitionist argument is in no way diluted by the destruction of these specific examples. I am making no substantive criticism of the intuitionist/constructivist position or of the humanist philosophy espoused by Hersh [5] and others. As effective computation and experimental mathematics—of which I am a passionate exponent—plays a larger and larger role in mathematics, it becomes less and less satisfactory to rely on the law of the excluded middle. The "quasi-empirical" nature of mathematics as argued for by Lakatos is more evident than ever. And as Hersh has suggested the philosophy of mathematics matters as never before.

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**REFERENCES**


