Structure of the correlation function at the accumulation points of the logistic map.

K. Karamanos\textsuperscript{a}, I.S. Mistakidis\textsuperscript{b}, S.I. Mistakidis\textsuperscript{c},

\textsuperscript{a}Department of Physics, University of Athens, Panepistimiopolis, GR 15784, Athens, Greece
kkaraman@ulb.ac.be
\textsuperscript{b}Hellenic Army Academy, Vari Attiki, Greece,
is.mist@hotmail.com
\textsuperscript{c}Department of Physics, University of Athens, Panepistimiopolis, GR 15784, Athens, Greece,
smistakidis@phys.uoa.gr

Abstract

The correlation function of the trajectory at the Feigenbaum attractors of the logistic map is rigorously introduced and checked by numerical experiments. Taking advantage of recent closed analytical results on the symbol-to-symbol correlation function, we are in position to justify the deep algorithmic structure of the correlation function apart from numerical constants. A generalization is given for arbitrary \( m \cdot 2^\infty \) Feigenbaum attractors.

Keywords:
Correlation function; Symbolic dynamics; Bifurcation points; Feigenbaum attractors; Logistic map.

1. Introduction

Recently, the Physics of Complex Systems has gained significant attention. One of the basic aspects of this progress is related with the understanding of correlations in and between such complex systems, which is realized through the use of different complexity measures. Among these, one can mention the transinformation, the block entropies and the correlation functions \[1-3,5,13,16-21,24,26-29,32-33\].

It is well-known the importance of correlation functions in the case of solid-state Physics and fluid mechanics for the detection of regularities in the spatial distribution of ions and atoms, especially in the cases of neutron scattering and short-laser pulses scattering. Also in kinetic theory, the correlation function of the momenta is connected with the derivation of the Maxwell-Boltzmann distribution function.

The correlation functions are then related to statistical ensembles and furthermore to Linear Response theory. The autocorrelation functions are also ultimately connected with Coding Schemes in Information Theory.

One of the Paradigms of Complex Systems is the logistic map. The logistic map has a simple definition but presents complex behavior when fine tuning the initial conditions and/or control parameter values. In particular, after Feigenbaum’s work, the period-doubling route to chaos has been fairly understood. Also, connections with the theory of second order phase transitions and
critical phenomena have been established. Scaling relations have been reported near to the accu-
accumulation point (also called Feigenbaum Point) with and without the presence of external noise
and cantorian fractal structures have been revealed in the transition point further connecting with
the physics of the non-chaotic attractor with self-similarity [4,7-12,14,30,33-34]. Recently also,
a direct connection to Experimental Mathematics has been established, too.
In Non-linear physics, the importance of the study of the correlation function has been realized
from the very beginning. Particularly inspiring have been the works of Ruelle [31], Daems and
Nicolis [6], and Alonso et al. [1], for the cases of resonances of chaotic dynamical systems. Based
on the analogies between the period doubling transition and critical phenomena Schuster has
done a guess on the functional form of the correlation function of the path [33]. According to
his arguments the correlation function should follow a power law behavior. In the present work
we propose some more detailed arguments, also based on the structure of the symbol-to-symbol
correlation function [20].

After establishing rigorously in a previous work the detailed form of the symbol-to-symbol cor-
relation function (that is the correlation function of symbolic dynamics) we turn now our study
in the structure of the correlation function of the trajectory. We shall show a Lemma of good
approximation of the correlation function of the trajectory by the symbol-to-symbol correlation
function. In a second stage we generalize these results for an arbitrary $m \cdot 2^n$ accumulation point.

The paper is organized as follows. In Sec.2 we introduce the logistic map and the definitions of
different types of correlation functions. In Sec.3 we present our careful numerical experimenta-
tion for the symbol-to-symbol correlation functions and for the correlation function of the traject-
ory at the (first) accumulation point. Those functions obey to simple numerical prescriptions,
which are explicitly outlined. A brief review of the analytic computation of the symbol-to-symbol
correlation function from first principles is also incorporated. We then present analogous results
and generalizations for the $m \cdot 2^n$ accumulation points. In Sec.4 we propose a Lemma of ”good
approximation” of the correlation function of the trajectory from the symbol-to-symbol correla-
tion function, which allows the justification of the functional form of the correlation function of
the trajectory apart from arithmetical constants in a systematic basis. Finally, in Sec.5 we draw
the main conclusions and discuss future works.

2. The logistic map

The logistic map is the archetype of a Complex System. Let us elaborate. We introduce the
logistic map in its familiar form:

$$x_{n+1} = rx_n(1 - x_n)$$  \hspace{1cm} (1)

For the logistic map in this form the generating partition is easily computed, following the
argument dating back to the French Mathematician Gaston Julia. To be more specific, for $f(x) =
rx \cdot (1 - x)$ the equation $f'(c) = 0$ gives $c=0.5$ so that the partition of the phase space (which
in this case is the unit interval I=[0,1]) $L=[0,0.5]$ and $R=(0.5,1]$ is a generating partition. The
information content of the symbolic trajectory is the ”minimum distinguishing information” in
the words of Metropolis et al. [25] Needless to say, in this representation the logistic map is
viewed as an abstract information generator.

In particular, the period doubling route to chaos has been fairly studied and it is by now well
understood. These studies led us to the occurrence of the two Feigenbaum constants $\alpha$ and $\delta$. Both
constants are defined by an approximate real space renormalization procedure. The constant $\delta$
has to do with the spacing in the control parameter space of the successive values of occurrence of the superstable periodic orbits and can be roughly estimated by the bifurcation diagram [9-10]. If we denote as \( \{r_n\} \) this set of values, \( \delta \) is defined as

\[
\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}
\]

(2)

and for the logistic map

\[
\delta \approx 4.669201609102990...
\]

(3)

The constant \( \alpha \) is related to the rescaling of the period doubling functional composition law and its value for the logistic map is

\[
\alpha = -\lim_{n \to \infty} \frac{d_n}{d_{n+1}} \approx -2.502907875095892...
\]

(4)

The two constants are not unrelated between them. A crude approximation gives

\[
\delta \approx \alpha^2 + \alpha + 1
\]

(5)

More accurate relations can be found by more refined renormalization group arguments. The values of the two constants depend only on the order of the maximum and have long been studied. They are thus, for instance, universal for quadratic maps irrespectively of the exact way one writes down the map.

In the above diagram the control parameter values of the bifurcation points are depicted as \( r_1, r_2, r_3, \ldots \) and the corresponding values for the superstable orbits are depicted as \( R_1, R_2, R_3, \ldots \). Feigenbaum and successors have shown that eq(2), holds if instead of \( r_i \) we put \( R_i \). Also, the values of \( d \), figuring in the definition of the Feigenbaum constant \( \alpha \) are noted in the figure 1.

We shall next define the correlation function of the trajectory as :

\[
C_{\omega m}(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{i+m} \chi_i, \text{ where } \chi_i = f^i(x_0) - \bar{x}, \quad \bar{x} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f^i(x_0)
\]

(6)
In direct analogy with the un-normalized correlation function we also introduce here the normalized correlation function:

\[ C(m) = \frac{C_{un}(m)}{C_{un}(0)} = \frac{C_{un}(m)}{\sigma^2} \]  

(7)

where \( \sigma \) is the mean standard deviation.

From these definitions follows that \( C(m) \) (or equally \( C_{un}(m) \)) yields another measure for the irregularity of the sequence of iterates \( x_0, f(x_0), f^2(x_0), \ldots \) It tells us how much the deviations of the iterates from their average value, \( \bar{x}_i = \bar{x} \) that are \( m \) steps apart (i.e., \( x_{i+m} \) and \( \bar{x}_i \)) "know" about each other, on the average. If \( C(m) \to 0 \) as \( m \to \infty \) then the system does not have the mixing property. The problem of determining the correlation function of an arbitrary dynamical system is intractable in the general case. This is the reason to resort to other computable observables such as the symbol-to-symbol correlation function [6].

So, in analogy with the correlation function of the trajectory we can introduce the symbol-to-symbol correlation function as:

\[ K_{un}(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \gamma_{i+m} \gamma_i \]  

(8)

where \( \gamma_i = 0,1 \) when \( x_i \leq 0.5 \) or \( x_i > 0.5 \) respectively and

\[ \gamma_i = y(f^i(x_0)) - \bar{y} \text{ where } \bar{y} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} y(f^i(x_0)) \]  

(9)

\[ r = FP \]

Figure 2: The symbol-to-symbol correlation function and the correlation function for the path is depicted for the logistic map exactly at the accumulation point \( r = FP \) and with initial condition \( x_0 = 0.5 \). The first \( 10^5 \) iterations have been eliminated in order to exclude transients, and the subsequent \( 10^8 \) iterations have been taken into account for the calculations. The experimentally determined Lyapunov is \(-5.93 \cdot 10^{-5}\)

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In the same way we can define the normalized symbol-to-symbol correlation function

\[
K(m) = \frac{K_{\text{un}}(m)}{K_{\text{un}}(0)} = \frac{K_{\text{un}}(m)}{\sigma^2} \quad (10)
\]

\[r = 3.60\]

Figure 3: The symbol-to-symbol correlation function and the correlation function for the path is depicted for the logistic map with \( r = 3.60 \), and with initial point \( x_0 = 0.5 \). The first \( 10^5 \) iterations have been eliminated in order to exclude transients, and the subsequent \( 10^8 \) iterations have been taken into account for the calculations. The distance from the accumulation point is \( \epsilon = 0.03 \) and the corresponding Lyapunov exponent is \( \lambda = 0.18 \). We observe an initial exponential envelope falling down as the exponential of the Lyapunov exponent.

3. Numerical experimentation

Motivated by previous work on correlation functions [6,31], we explore here numerically the properties of the symbol-to-symbol correlation function and the correlation function of the path. In order to cope with the problem of the analytic form of correlation functions we stalled certain number of numerical experiments. For the logistic map at the Feigenbaum point \( r = 3.56994567 \ldots \) we have calculated both the normalized symbol-to-symbol correlation function and the correlation function of the trajectory for \( 10^8 \) iterations (we eliminated the first \( 10^5 \) iterations to avoid transients), starting from the initial point \( x_0 = 0.5 \). As also stated in [10], exactly at the Feigenbaum point the Lyapunov exponent strictly vanishes \( \lambda = 0 \).

In a previous work by the same authors [20], an empirical rule for the un-normalized symbol-to-symbol correlation function has been established after careful numerical experimentation, namely that:

\[
K_{\text{un}}(m) = A_r \cdot \delta_{m,2^{-1}(1+2k)} \quad (11)
\]
Figure 4: The symbol-to-symbol correlation function and the correlation function for the trajectory is depicted for the logistic map with $r = 3.8495$ (cycle $3 \cdot 2^n$) and with initial condition $x_0 = 0.5$. The first $10^3$ iterations have been eliminated in order to exclude transients, and the subsequent $10^8$ iterations have been taken into account for the calculations.

where for a given (fixed) $r$, $r \in \{1, 2, 3, \ldots\}$, $A_r$ is a constant depending only on $r$, and $k$ takes all the values from the set of natural numbers $\{0, 1, 2, 3, \ldots\}$.

We also notice here that according to our computations, for $m \geq 256$ the correlation function $(K_{un}(m))$ approaches the value $2/9$.

Furthermore, on the basis of the Metropolis-Stein-Stein algorithm, we can justify the functional form of equation (11), and calculate analytically by an algorithmic procedure constructively (that is step-by-step [21]) the constant coefficients $A_r$, finding for the first terms:

$$K_{un}(m) = \begin{cases} \frac{-1}{9}, & m \text{ odd} = 1 + 2 \cdot k \quad (k = 0, 1, 2, \ldots) \\ \frac{1}{10}, & m = 2 + 4 \cdot k \quad (k = 0, 1, 2, \ldots) \\ \frac{1}{11}, & m = 4 + 8 \cdot k \quad (k = 0, 1, 2, \ldots) \\ \frac{1}{12}, & m = 8 + 16 \cdot k \quad (k = 0, 1, 2, \ldots) \\ \frac{1}{13}, & m = 16 + 32 \cdot k \quad (k = 0, 1, 2, \ldots) \end{cases} \quad (12)$$

We can extend this procedure to infinity. The constructive scheme guarantees that this deep algorithmic structure is kept in all scales. From this infinite stratification, the infinite memory of the system at the Feigenbaum point is revealed, as this scheme never ends.

We now extend our numerical experimentation to the structure of the normalized symbol-to-symbol correlation function for $10^8$ iterations (this scheme is depicted in Fig.2, and for a small deviation from the accumulation point see Fig.3):
Figure 5: The symbol-to-symbol correlation function and the correlation function for the trajectory is depicted for the logistic map with \( r = 3.7430 \) (cycle \( 5 \cdot 2^\infty(a) \)) and with initial condition \( x_0 = 0.5 \). The first 10\(^5\) iterations have been eliminated in order to exclude transients, and the subsequent 10\(^8\) iterations have been taken into account for the calculations.

\[
K(m) = \begin{cases} 
1, & m - \text{odd} = 1 + 2 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/8, & m = 2 + 4 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/16, & m = 4 + 8 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/32, & m = 8 + 16 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/64, & m = 16 + 32 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/128, & m = 32 + 64 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/256, & m = 64 + 128 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/512, & m = 128 + 256 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/1024, & m = 256 + 512 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/2048, & m = 512 + 1024 \cdot k \quad (k = 0, 1, 2, \ldots) 
\end{cases}
\]

This structure has been established in [20]. Moreover, for the correlation function of the trajectory we find:

\[
C(m) = \begin{cases} 
-1/16, & m - \text{odd} = 1 + 2 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/16, & m = 2 + 4 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/32, & m = 4 + 8 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/64, & m = 8 + 16 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/128, & m = 16 + 32 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/256, & m = 32 + 64 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/512, & m = 64 + 128 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/1024, & m = 128 + 256 \cdot k \quad (k = 0, 1, 2, \ldots) \\
1/2048, & m = 256 + 512 \cdot k \quad (k = 0, 1, 2, \ldots) 
\end{cases}
\]
We observe that it has also the functional form of equation (11) apart from numerical constants $A'_p$. It is clear that the Lemma (1) predicts correctly the functional form of the correlation function.

Table 1: In the second column of the table below we present the values of the control parameter corresponding to the different accumulation points of the logistic map. In the third and fourth columns they are presented the Lyapunov exponents for $10^8$ iterations, including and excluding transients correspondingly. We observe no significant differences due to the augmented statistics. 

<table>
<thead>
<tr>
<th>Accumulation cycle</th>
<th>Accumulation point</th>
<th>Lyapunov exponent $\pm 10^{-5}$</th>
<th>Lyapunov exponent $\pm 10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^\infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3 \cdot 2^m$</td>
<td>3.8495</td>
<td>$-5.934 \cdot 10^{-5}$</td>
<td>$-5.934 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$4 \cdot 2^m$</td>
<td>3.9612</td>
<td>0.0237</td>
<td>0.0237</td>
</tr>
<tr>
<td>$5 \cdot 2^m(a)$</td>
<td>3.7430</td>
<td>$-0.0021$</td>
<td>$-0.0021$</td>
</tr>
<tr>
<td>$5 \cdot 2^m(b)$</td>
<td>3.9065</td>
<td>0.0414</td>
<td>0.0414</td>
</tr>
<tr>
<td>$5 \cdot 2^m(c)$</td>
<td>3.99032</td>
<td>$-0.0039$</td>
<td>$-0.0039$</td>
</tr>
<tr>
<td>$6 \cdot 2^m(a)$</td>
<td>3.6327</td>
<td>$-0.0073$</td>
<td>$-0.0073$</td>
</tr>
<tr>
<td>$6 \cdot 2^m(b)$</td>
<td>3.937649</td>
<td>0.0127</td>
<td>0.0127</td>
</tr>
<tr>
<td>$6 \cdot 2^m(c)$</td>
<td>3.977800</td>
<td>0.0077</td>
<td>0.0077</td>
</tr>
<tr>
<td>$6 \cdot 2^m(d)$</td>
<td>3.997586</td>
<td>$-0.0133$</td>
<td>$-0.0133$</td>
</tr>
</tbody>
</table>

Proceeding further to the $3 \cdot 2^m$ scenario which corresponds to the control parameter value $r=3.8495$ (see Table 1) we find:

$$C(\tau) = A_r \cdot \tau^{1.2 - (1+2k)} + B_1 \cdot \tau^{1.2 - (1+2k)} + B_2 \cdot \tau^{1.2 - (1+2k)}$$  \hspace{1cm} (15)$$

Table 2: In the table below we present the first few coefficients of eq(16) for the normalized symbol-to-symbol correlation function for each cycle and the corresponding mean value for which we have taken into account the first $10^8$ iterations of the map.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>mean value</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^m$</td>
<td>0.6666</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$3 \cdot 2^m$</td>
<td>0.4381</td>
<td>1.1702</td>
<td>1.1702</td>
<td>1.1702</td>
<td>1.1702</td>
<td>1.1702</td>
<td>1.1702</td>
<td>1.1702</td>
<td>1.1702</td>
</tr>
<tr>
<td>$4 \cdot 2^m$</td>
<td>0.3342</td>
<td>1.6483</td>
<td>1.6483</td>
<td>1.6483</td>
<td>1.6483</td>
<td>1.6483</td>
<td>1.6483</td>
<td>1.6483</td>
<td>1.6483</td>
</tr>
<tr>
<td>$5 \cdot 2^m(a)$</td>
<td>0.6625</td>
<td>1.1026</td>
<td>1.1026</td>
<td>1.1026</td>
<td>1.1026</td>
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<td>1.1026</td>
<td>1.1026</td>
<td>1.1026</td>
</tr>
<tr>
<td>$5 \cdot 2^m(b)$</td>
<td>0.5420</td>
<td>1.2949</td>
<td>1.2949</td>
<td>1.2949</td>
<td>1.2949</td>
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<td>1.2949</td>
<td>1.2949</td>
<td>1.2949</td>
</tr>
<tr>
<td>$5 \cdot 2^m(c)$</td>
<td>0.2500</td>
<td>1.3636</td>
<td>1.3636</td>
<td>1.3636</td>
<td>1.3636</td>
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<tr>
<td>$6 \cdot 2^m(a)$</td>
<td>0.7708</td>
<td>1.5775</td>
<td>1.5775</td>
<td>1.5775</td>
<td>1.5775</td>
<td>1.5775</td>
<td>1.5775</td>
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</tr>
<tr>
<td>$6 \cdot 2^m(b)$</td>
<td>0.5529</td>
<td>1.2785</td>
<td>1.2785</td>
<td>1.2785</td>
<td>1.2785</td>
<td>1.2785</td>
<td>1.2785</td>
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</tr>
<tr>
<td>$6 \cdot 2^m(c)$</td>
<td>0.4468</td>
<td>1.2744</td>
<td>1.2744</td>
<td>1.2744</td>
<td>1.2744</td>
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<td>1.2744</td>
<td>1.2744</td>
<td>1.2744</td>
</tr>
<tr>
<td>$6 \cdot 2^m(d)$</td>
<td>0.2500</td>
<td>1.511</td>
<td>1.511</td>
<td>1.511</td>
<td>1.511</td>
<td>1.511</td>
<td>1.511</td>
<td>1.511</td>
<td>1.511</td>
</tr>
</tbody>
</table>

where the first few numerical values of the above coefficients are presented in Table 2 for the correlation function of the symbolic sequence. The same structure is observed for the correlation function of the trajectory, apart from numerical constants $A'_p, B'_1, B'_2$ which depend on the detailed form of the map. In Table 3 we extend these considerations to the $4 \cdot 2^m, 5 \cdot 2^m(a), 5 \cdot 2^m(b), 5 \cdot 2^m(c)$. 

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information about the accumulations points and the corresponding patterns has been found in
As a result of the above studies we conclude that the suggested general form of the correlation
function of the order:
\[ 4. \text{The good approximation Lemma} \]

\[ \text{Lemma 1.} \] The correlation function of the trajectory (as defined in eq(6)) contains the same
time scales with the symbol-to-symbol correlation function (eq(8)). That is:
\[ C_{\text{ms}}(m) = A_r' \cdot \delta_{m2^{-i}(1+2k)} \]  

\[ (0.5 + \varepsilon_i) \cdot (0.5 + \varepsilon_j) \approx 0.25 + \varepsilon + O(\varepsilon^2) \]  

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that is more than 0.25.

- A contribution in the symbolic correlation function by two terms $x_k < 0.5$ and $x_{k+r} < 0.5$ is 0 and gives a corresponding contribution to the trajectory correlation function of the order:

\[(0.5 - \varepsilon_i) \cdot (0.5 - \varepsilon_j) \approx 0.25 - \varepsilon + O(\varepsilon^2) \]  \hspace{1cm} (19)

that is less than 0.25.

- A contribution in the symbolic correlation function by two terms $x_k > 0.5$ and $x_{k+r} < 0.5$ is 0 and gives a corresponding contribution to the trajectory correlation function of the order:

\[(0.5 - \varepsilon_i) \cdot (0.5 - \varepsilon_j) \approx 0.25 + O(\varepsilon^2) \]  \hspace{1cm} (20)

that is of the order of 0.25.

- A contribution in the symbolic correlation function by two terms $x_k < 0.5$ and $x_{k+r} > 0.5$ is 0 and gives a corresponding contribution to the trajectory correlation function of the order:

\[(0.5 - \varepsilon_i) \cdot (0.5 - \varepsilon_j) \approx 0.25 + O(\varepsilon^2) \]  \hspace{1cm} (21)

that is of the order of 0.25.

In order to clarify the meaning of the above approximations let us consider a specific example: Let $x_k = 0.7$ and $x_{k+r} = 0.9$. Then $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.4$. So, the contribution to the symbolic correlation function is 1 and as a consequence the contribution to the real correlation function is $0.63 = 0.25 + \varepsilon > 0.25$.

5. Conclusions

The correlation function is an important quantity measuring correlations in many branches of physics. Obviously, there are also other interesting quantities as for instance the (conditional) block-entropies, the transinformation, the Kolmogorov-Sinai entropy etc. However, it does provide an important measure of correlations by itself.

In a first step the correlation function of the trajectory at the Feigenbaum point is numerically investigated with careful numerical experimentation. We arrived to an empirical formula summarizing the results.

Comparing with the symbol-to-symbol correlation function discovered in the literature theoretically and numerically we observe that it contains the same time scales, that is, it has the same functional form.

This result is justified on the basis of a good approximation Lemma, introduced for the first time in the present paper. Moreover, we generalize these results for the case of an arbitrary $m \cdot 2^\infty$ Feigenbaum non chaotic multifractal attractor. A comparison with numerical results is also included.

To recapitulate, we are in position to justify the analytical form of the correlation function of the trajectory from first principles (the MSS algorithm) and in a systematic way, apart from numerical constants dependent on the detailed functional form of the map. Apart from their mathematical beauty such ideas find important practical applications ranging from preisismic signals [15] to DNA sequence analysis [2,23], Heart beat rythms [22] and Linguistics Processes.