Phase retrieval, Gerchberg-Saxton algorithm, and Fienup variants:  
A view from convex optimization

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Abstract

The phase retrieval problem is of paramount importance in various areas of applied physics and engineering. The state of the art for solving this problem in two dimensions relies heavily on the pioneering work of Gerchberg, Saxton, and Fienup. Despite the widespread use of the algorithms proposed by these three researchers, current mathematical theory cannot explain their remarkable success. Nevertheless, great insight can be gained into the behavior, the shortcomings, and the performance of these algorithms from their possible counterparts in convex optimization theory. An important step in this direction was made two decades ago when the Gerchberg-Saxton algorithm was identified as a nonconvex alternating projection algorithm. The purpose of this paper is to formulate the phase retrieval problem with mathematical care and to establish new connections between well established numerical phase retrieval schemes and classical convex optimization methods. Specifically, it is shown that Fienup’s basic input-output algorithm corresponds to Dykstra’s algorithm, and that Fienup’s hybrid input-output algorithm can be viewed as an instance of the Douglas-Rachford algorithm. This work provides a theoretical framework to better understand and, potentially, improve existing phase recovery algorithms.

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1 Introduction

The phase retrieval problem consists of estimating the phase of a complex-valued function from measurements of its modulus and additional a priori information. It is of fundamental importance in numerous areas of applied physics and engineering [5, 16, 29, 64, 66, 56, 72, 80] and has been studied for over forty years (see [70, 84, 85] and the references therein). Historically, the roots of the problem can be traced back to 1892: in a letter to A. Michelson, Lord Rayleigh stated that the continuous phase retrieval problem in interferometry was in general impossible to solve without a priori information on the symmetry of the data [76].

As in many inverse problems, a common formulation of the phase retrieval problem is to seek as a solution any function that is consistent with the measurements as well as with a priori constraints. The Gerchberg-Saxton algorithm [44] was the first widely used numerical scheme to solve this type of problem. While its intrinsic mechanism is clear physically — it consists of alternating back-substitutions of known information in the spatial and Fourier domains — it was not initially understood mathematically. In particular, failure of convergence and stagnation of the iterates away from solution points were observed from the outset but lacked a sound mathematical explanation. In the early 1980s, with the work of Youla [88] and others [57, 58, 82], the application of Bregman’s method of successive projections [14] to the recovery a signal described by convex constraints generated considerable interest in the signal recovery community; see [21, 23, 74, 75] and the extensive lists of references therein. It was natural to seek to embed other iterative methods in this powerful projection framework. Thus, the informal use of cyclic projections in the presence of nonconvex sets appears in several places in the literature [27, 50, 52, 71]. In [59], the Gerchberg-Saxton algorithm was revealed as such an algorithm, featuring a nonconvex magnitude constraint in the underlying signal space. This study (see also the follow-up paper [60]) gave insightful geometrical interpretations of the stagnation problem as well as of other aspects of the Gerchberg-Saxton iterative procedure. A local convergence statement for the general nonconvex projection method was then proposed in [17] and followed in [28] by a more formal analysis based on the theory of multi-valued projections (see also [54] for a tutorial review of these two papers, and [20] for further developments). Another approach to the convergence question was proposed in [3, 4], based on the projection theory for convex sets. To extend this theory to the nonconvex setting, the authors require the projection operators to be single-valued\(^1\). Unfortunately, there is no known example of a nonconvex set for which the projection operator is single-valued (see Remark 3.10). Indeed, the projections in the phase retrieval problem are inherently multi-valued: in [62], the projection operator is precisely identified with the multi-valued subdifferential of a related nonsmooth error metric. A smooth approximation to the projection operator is presented in [63], together with results on the local convergence of iterative methods for minimizing a corresponding smooth error metric.

In a series of papers [36, 37, 38] which were unified and reviewed in his seminal 1982 paper [39], Fienup introduced a broad framework for iterative algorithms. Three main classes of algorithms were presented: Output-Output (which covers the Gerchberg-Saxton algorithm), Basic Input-Output

\(^1\) The issue of nonuniqueness of the projection is not to be confused with the uniqueness of of solutions to the phase problem. The results surveyed, for instance, in [49] are not effected by the multi-valuedness of the projection operators.
(BIO), and Hybrid Input-Output (HIO). This work resulted in new applications in a wide range of imaging modalities. Furthermore, the Gerchberg-Saxton and Fienup algorithms continue to constitute the state of the art in phase retrieval.

While widely accepted by practitioners, the BIO and HIO algorithms lack a proper mathematical framework. The aim of this paper is to show that, just like the Gerchberg-Saxton algorithm, the BIO and HIO algorithms also have well-known and powerful counterparts in the world of convex projection methods. Our discussion is somewhat more formal than what is usually found in the optics literature, as the level of sophistication of the algorithms requires great attention to details. As much as possible, we provide an intuitive and non-technical discussion without sacrificing the mathematical rigor and precision necessary for a meaningful and constructive analysis of Fienup’s algorithms. All technical proofs have been relegated to an appendix.

The paper is organized as follows. In Section 2, the phase retrieval problem is posed as a feasibility problem. Section 3 supplies the necessary review of projection theory, convex analysis, and fixed point theory. The classical algorithms for solving the phase retrieval problem are presented in Section 4. Section 4.4 describes the formal correspondence between these algorithms and classical algorithms for solving the convex optimization problems: Gerchberg-Saxton and alternating projections (Section 4.4.2); Fienup’s BIO and Dykstra’s algorithm (Section 4.4.3); Fienup’s HIO and Douglas-Rachford (Section 4.4.4). Concluding remarks are formulated in Section 5.

2 Phase retrieval and feasibility

In its general form, the signal recovery problem is to estimate the original form of a signal $x$ in a functional space $L$ from the measurements of physically related signals and a priori information [23, 74]. In phase retrieval problems, the measurements consist of the modulus $m$ of the Fourier transform $\hat{x}$ of $x$. In other words, the imaging model is described by the relationship

$$|\hat{x}| = m,$$

and $x$ is commonly referred to as the object or input of the imaging model. For instance, in optical interferometry astronomy, $x$ is the scattering amplitude of some medium [55, Chapter 2]. The measurement, $m : \mathbb{R}^2 \to \mathbb{R}_+$, is proportional to the modulus of the spatial coherence function [46, Section 7.4], that is, $m$ is proportional to the modulus of the Fourier transform of the scattering amplitude.

A general signal space that appropriately models the underlying physics is the complex Hilbert space

$$L = L^2[\mathbb{R}^N, \mathbb{C}].$$

Hence, a signal $x$ in $L$ is a square-integrable function mapping a continuous variable $t \in \mathbb{R}^N$ to a complex number $x(t) \in \mathbb{C}$. The set of signals that satisfy the image domain constraint (1) is\(^2\)

$$M = \{ y \in L : |\hat{y}| = m \text{ a.e.}\}.$$

\(^2\)“a.e.” stands for “almost everywhere” in the sense of measure theory since, strictly speaking, the elements of $L$
In addition to the imaging model, an important piece of information that is typically available in phase retrieval problems is that the support of $x$ is contained in some set $D \subset \mathbb{R}^N$. If we let $1_E$ denote the characteristic function of a set $E \subset \mathbb{R}^N$ and $\mathbf{C} \mathcal{E}$ its complement, this \textit{object domain constraint} confines $x$ to the set
\[ S = \{ y \in \mathcal{L} : y \cdot 1_{\mathbb{C} D} = 0 \}. \tag{4} \]

The \textit{phase retrieval problem} can be posed as that of finding a function $x \in \mathcal{L}$ that satisfies these two constraints, namely,
\[ \text{find} \quad x \in S \cap M. \tag{5} \]

This formulation exhibits the phase retrieval problem as a problem of finding a point in the intersection of constraint sets, i.e., a \textit{set theoretic estimation problem} in the sense of [21]. In mathematics (especially in optimization) problems of this kind are called \textit{feasibility problems}. In this paper we shall restrict our attention the case when (5) is consistent, i.e., $S \cap M = \emptyset$. It should be noted, however, that occurrences of inconsistent set theoretic formulations in phase retrieval or other signal recovery problems are far from being academic due to noisy data, measurement errors, or inaccurate \textit{a priori} information [19, 22, 23, 51, 73]. Several investigations have been devoted to analyzing and coping with this situation in convex problems [6, 8, 11, 26, 47, 87].

While the infinite-dimensional space $\mathcal{L}$ appears to be the most appropriate signal space to model the physics of the problem and to describe the subtle properties of the algorithms in their full generality, we shall also call attention to finite-dimensional versions of the results whenever these happen to differ from their infinite-dimensional counterparts. The reason for this is that in most numerical applications, the signals are sampled on a finite grid and the algorithms are implemented on a digital computer\textsuperscript{3}. In this context, the underlying Hilbert space is the usual $p$-dimensional Euclidean space, where $p$ is the number of samples in the discretized signal.

\section{Fundamentals of numerical theory}

Before discussing the most common and successful algorithms for solving the phase retrieval problem, we establish the mathematical definitions, properties, and results that constitute the theoretical foundation of projection algorithms. We begin with a few basic definitions.

\subsection{Distances, projections, and projectors}

\subsubsection{Distances}

As we shall deal with different Hilbert spaces, we assume in this section that

\footnotetext{\textsuperscript{3} Iterative signal recovery projection algorithms have also been implemented optically without sampling the continuous waveforms, e.g., [65]. In such instances, the underlying signal space is $\mathcal{L}$ itself.}
\( \mathcal{H} \) is a general Hilbert space, with inner product \( \langle \cdot, \cdot \rangle \),

and corresponding norm \( \|x\| = \sqrt{\langle x, x \rangle} \) for \( x \in \mathcal{H} \).

For instance, if \( \mathcal{H} = \mathbb{L} \), then \( \langle x, y \rangle = \int x \overline{y} \), for \( x, y \in \mathcal{H} \). Or \( \langle x, y \rangle = x^T y \) in \( \mathbb{R}^N \). The quantity \( \|x\|^2 \) is simply the energy of a signal \( x \in \mathcal{H} \).

**Definition 3.1 (distances)** Suppose \( x \in \mathcal{H} \).

(i) If \( y \) is a point in \( \mathcal{H} \), then the distance from \( x \) to \( y \) is

\[
d(x, y) = \| x - y \|.
\]

(ii) If \( Y \) is a set in \( \mathcal{H} \), then the distance from \( x \) to \( Y \) is\(^4\)

\[
d(x, Y) = \inf_{y \in Y} d(x, y).
\]

The distance from a point to a set may not be attained:

**Example 3.2** Let \( \mathcal{H} = \mathbb{R}^2 \) be the Euclidean plane. Then:

(i) If \( x = (2, 0) \) and \( Y = \{ y \in \mathcal{H} : \| y \| \leq 1 \} \) is the unit ball, then \( d(x, Y) = 1 \) and the distance is attained at \( y = (1, 0) \in Y: d(x, Y) = d(x, y) \). Moreover, \( y \) is the only point in \( Y \) with this property — every other point in \( Y \) is further away from \( x \) than \( y \) is.

(ii) If \( x = (2, 0) \) and \( Y = \{ y \in \mathcal{H} : \| y \| < 1 \} \) is the open unit ball, then still \( d(x, Y) = 1 \) yet there is no point \( y \in Y \) with \( d(x, y) = d(x, Y) \); the distance from \( x \) to \( Y \) is not attained.

(iii) If \( x = (0, 0) \) and \( Y = \{ y \in \mathcal{H} : \| y \| = 1 \} \) is the unit circle, then \( d(x, Y) = 1 = d(x, y) \), for all \( y \in Y \); the distance is attained at every point in \( Y \).

The points at which the distance to a set is attained are of great importance and the subject of the following subsection.

### 3.1.2 Projection operators (projectors)

**Definition 3.3 (projection operator)** Suppose \( Y \) is a set in \( \mathcal{H} \). If \( x \in \mathcal{H} \), then the set of points in \( Y \) nearest to \( x \), namely

\[
\{ y \in Y : d(x, y) = d(x, Y) \},
\]

is denoted \( P_Y(x) \), and called the projection of \( x \) onto \( Y \). The induced operator \( P_Y \) is called the projection operator or projector onto \( Y \).

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\(^4\)Let \( R \) be a set of real numbers. If \( R \neq \emptyset \), then “\( \inf(R) \)” stands for the infimum of \( R \), i.e., the largest number in \( [-\infty, +\infty] \) that is smaller than or equal to all elements of \( R \). By convention, \( \inf\emptyset = +\infty \).
An interpretation from signal processing helps at this point: if \( Y \) contains the signals satisfying a certain property, then the signals in \( P_Y(x) \) are the closest signals to \( x \) satisfying this property. It is crucial to realize that the output of a projector \( P_Y \) are subsets of \( Y \). These may be empty, reduced to a single element, or have more than one element: revisiting Example 3.2 and borrowing its notation, we see that (i) \( P_Y(x) = \{(1,0)\} \), (ii) \( P_Y(x) = \emptyset \), (iii) \( P_Y(x) = Y \), respectively. To bring out this behavior clearly, we say that the projector is a multifunction or a multi-valued map [2].

**Remark 3.4 (single-valued projections)** Suppose the projection of \( x \) onto a set \( Y \) is a singleton, say \( P_Y(x) = \{y\} \) for some \( y \in Y \). It is common practice to simply write \( P_Y(x) = y \), a slight abuse of notation. This always occurs when the set \( Y \) is closed and convex, see Fact 3.9 below.

### 3.1.3 A preview: Projections for the phase retrieval problem

In the setting of the phase retrieval problem, the abstract Hilbert space \( \mathcal{H} \) is simply the function space \( \mathcal{L} \) introduced in Section 2. The most common approach for solving the phase retrieval problem is to enforce the known object domain and image domain constraints in some alternating fashion. Thus, given a signal \( x \), the support constraint is naturally enforced by setting \( x \) equal to zero outside the given domain \( D \), via the transformation \( x \mapsto x \cdot 1_D \). As we shall now see, this simple operation is actually a projection:

**Example 3.5 (support constraint)** Suppose \( D \) is a measurable\(^5\) set in \( \mathbb{R}^N \) and fix \( x \in \mathcal{L} \). Then the projection (recall the convention of Remark 3.4) of \( x \) onto the set \( S \) of (4) is

\[
P_S(x) = x \cdot 1_D.
\]

(7)

The same observation is true for the image modulus constraint. Approaches to enforce it are described below; again, these operations turn out to be projections.

**Example 3.6 (image modulus constraint)** Let \( m \) be a nonnegative function in \( \mathcal{L} \) and fix \( x \in \mathcal{L} \). Then \( y \in \mathcal{L} \) belongs to the projection \( P_Y(x) \) of \( x \) onto the set \( M \) of (3) if and only if it satisfies a.e.

\[
\hat{y}(\omega) = \begin{cases} 
m(\omega) \frac{\hat{x}(\omega)}{|\hat{x}(\omega)|}, & \text{if } \hat{x}(\omega) \neq 0; \\
m(\omega) \exp[i\varphi(\omega)], & \text{otherwise},
\end{cases}
\]

(8)

for some measurable function \( \varphi : \mathbb{R}^N \to \mathbb{R} \).

\(^5\)For theoretical reasons, the sets (and functions) we deal with must be “measurable” — this is not the same “physically measurable” or “observable”! For our purposes, measurable sets and functions constitute a sufficiently large class to work with; thus, all closed and open subsets (and all continuous functions) are measurable as well as various combinations of those. See, for instance, [1, Chapter 4] for further information.
Example 3.6 shows that every function \( y \in P_Y(x) \) satisfies
\[
d(\widehat{x}(\omega), m(\omega)\mathbb{S}) = d(\widehat{x}(\omega), \widehat{y}(\omega)) \quad \text{a.e. on } \mathbb{R}^N, \tag{9}\]
where \( m(\omega)\mathbb{S} = \{ u \in \mathbb{C} : |u| = m(\omega) \} \) denotes a circle in the complex plane, with radius \( m(\omega) \) and centered at the origin. The \textit{multi-valuedness} of the projection is now evident: whenever \( \widehat{x}(\omega) = 0 \), any phase \( \varphi \) will work. Consequently, if the set \( \{ \omega \in \mathbb{R}^N : m(\omega) \neq 0 \text{ and } \widehat{x}(\omega) = 0 \} \) is sufficiently large\(^6\), then \( P_Y(x) \) contains \textit{infinitely many} elements (see [63] and Example 3.16 below).

In practice, one picks the \textit{particular selection} \( y_0 \in P_Y(x) \) corresponding to zero phase \( \varphi \equiv 0 \):
\[
\widehat{y}_0(\omega) = \begin{cases} m(\omega) \frac{\widehat{x}(\omega)}{|\widehat{x}(\omega)|}, & \text{if } \widehat{x}(\omega) \neq 0; \\ m(\omega), & \text{otherwise}. \end{cases} \tag{10}\]

Analogous formulae hold if one considers a modulus constraint in the object domain (as in the original set-up of the Gerchberg-Saxton algorithm for reconstructing phase from two intensity measurements; see [44]).

### 3.2 Convexity and closedness

In what follows we assume that\(^7\)
\[
\mathcal{H} \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle \text{ and induced norm } \| \cdot \|. \tag{11}\]

**Definition 3.7 (convex set)** Suppose \( C \) is a set in \( \mathcal{H} \). Then \( C \) is \textit{convex}, if the line segment joining any two points in \( C \) lies entirely in \( C \); algebraically: \( \lambda_1 c_1 + \lambda_2 c_2 \in C \) whenever \( c_1, c_2 \in C \), and \( \lambda_1, \lambda_2 \geq 0 \) with \( \lambda_1 + \lambda_2 = 1 \).

**Definition 3.8 (closed set)** Suppose \( C \) is a set in \( \mathcal{H} \). Then \( C \) is \textit{closed}, if whenever \( (c_n) \) is a sequence in \( C \) that converges to some point \( x \in \mathcal{H} \), then this limit point \( x \) belongs to \( C \).

Closedness is important for algorithmic purposes: one often wants the limit of a sequence to inherit good properties from the terms of the sequence. Closedness is certainly a necessary condition for the existence of projections. Indeed, a point \( x \) in the closure (smallest closed superset) of \( C \) but not in \( C \) has no projection onto \( C \). In finite-dimensional spaces, closedness is also sufficient to guarantee the existence of projections, e.g., [28]; however, this is no longer true in infinite-dimensional spaces (see [81, Example III.4.3.2.b] for a counterexample).

In tandem with convexity, closedness guarantees that projections are extremely well-behaved:

\(^6\)Mathematically, this set is assumed to have nonzero measure.

\(^7\)The complex Hilbert space \( \mathcal{L} \) from the phase retrieval problem is also a real Hilbert space provided that we use the real part of the inner product as the new inner product.
Fact 3.9 (projection onto a closed convex set) Suppose $C$ is a nonempty closed convex set in $\mathcal{H}$. Then for every $x \in \mathcal{H}$, the projection of $x$ onto $C$ is a singleton; moreover, the point$^8$ $P_C(x)$ is characterized by

$$P_C(x) \in C, \quad \text{and} \quad (c - P_C(x), x - P_C(x)) \leq 0, \quad \text{for all} \ c \in C. \quad (12)$$

In addition, the projector $P_C$ satisfies

$$\|P_C(x) - P_C(y)\|^2 + \|(I - P_C)(x) - (I - P_C)(y)\|^2 \leq \|x - y\|^2, \quad \text{for all} \ x, y \in \mathcal{H}. \quad (13)$$

Remark 3.10 (Chebyshev problem) Suppose $C$ is a closed nonempty set in $\mathcal{H}$. If $C$ is convex, then Fact 3.9 states that the projector $P_C$ is a single-valued map. The converse implication is the famous Chebyshev problem: if the projector $P_C$ is a single-valued map, must the set $C$ be convex? The answer is affirmative in finite-dimensional spaces, but remains open to date for the general Hilbert space case. If it turns out to be affirmative in general, then the results of [3, 4] discussed in the Introduction are essentially void. The reader is referred to [31, Chapter 12] for further information.

For the remainder of this section, we use the notation of Section 2. The following result is quite useful.

Proposition 3.11 (separable constraints and projections) Suppose $(A(t))_{t \in \mathbb{R}^N}$ is a family of sets in $\mathcal{C}$. Let $\mathcal{A} = \{a \in \mathcal{L} : a(t) \in A(t) \text{ a.e.}\}$. Then:

(i) If each $A(t)$ is convex, then so is $\mathcal{A}$.

(ii) If each $A(t)$ is closed, then so is $\mathcal{A}$.

Theorem 3.12 (projection onto a separably closed convex constraint) Let $(A(t))_{t \in \mathbb{R}^N}$ be a family of closed convex sets in $\mathcal{C}$ such that $\mathcal{A} = \{a \in \mathcal{L} : a(t) \in A(t) \text{ a.e.}\}$ is nonempty. Assume that $t \mapsto A(t)$ is a measurable multifunction$^9$ from $\mathbb{R}^N$ to $\mathcal{C}$. Fix $x \in \mathcal{L}$. Then $y = P_{\mathcal{A}}(x)$ is given by $y(t) = P_{A(t)}(x(t))$ a.e.

Remark 3.13 In Theorem 3.12, the condition that $t \mapsto A(t)$ be measurable may be difficult to verify in practice. For our purpose, however, it is sufficient to work with the following criterion, taken from [69, Section 14.A]:

if $\mu$ is a measurable function from $\mathbb{R}^N$ to $\mathcal{C}$ and $Z \subseteq \mathcal{C}$, then $A(t) = \mu(t) \cdot Z$ defines a measurable multifunction.

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$^8$Recall the convention from Remark 3.4.

$^9$In our setting, this means that the set $\{t \in \mathbb{R}^N : A(t) \cap Z \neq \emptyset\}$ is measurable, for every closed (or, equivalently, open) set $Z$ in $\mathcal{C}$; see [2, Section 8.1] and [69, Section 14.A].
Note that there is no restriction whatsoever on the set $Z$.

**Example 3.14 (object domain constraint)** Let $x \in \mathcal{L}$. Then:

(i) (support) $\mathcal{A} = \{y \in \mathcal{L} : y \cdot 1_{\mathcal{C} D} = 0\}$ is closed and convex, with $P_{\mathcal{A}}(x) = x \cdot 1_D$.

(ii) (real-valuedness) $\mathcal{A} = \{y \in \mathcal{L} : y(t) \in \mathbb{R} \text{ a.e.}\}$ is closed and convex, with $P_{\mathcal{A}}(x) = \Re(x)$.

(iii) (nonnegativity) $\mathcal{A} = \{y \in \mathcal{L} : y(t) \text{ is real and nonnegative a.e.}\}$ is closed and convex, with $^{10} P_{\mathcal{A}}(x) = (\Re(x))^+$.

**Theorem 3.15 (projection onto a separably compact constraint)** Let $(A(t))_{t \in \mathbb{R}^N}$ be a family of compact (i.e., closed and bounded) sets in $\mathbb{C}$ such that $\mathcal{A} = \{a \in \mathcal{L} : a(t) \in A(t) \text{ a.e.}\}$ is nonempty. Assume that $t \mapsto A(t)$ is a measurable multifunction from $\mathbb{R}^N$ to $\mathbb{C}$. Then, for every $x \in \mathcal{L}$, $P_{\mathcal{A}}(x) \neq \emptyset$; in fact, $y \in P_{\mathcal{A}}(x)$ if and only if $y$ is measurable and $y(t) \in P_{A(t)}(x(t))$ a.e.

**Example 3.16 (image domain constraint is closed but not convex)** For closedness, see the proof in Appendix A. Unless $m = 0$ (in which case the image modulus constraint encompasses only the zero function), the image modulus constraint is never a convex set. To see this, pick $x \in M$. Then $-x \in M$; however, the convex combination $\frac{1}{2}x + \frac{1}{2}(-x) = 0$ does not belong to $M$.

While the phase retrieval problem (5) is not convex, some related problems which are convex can be found in the literature.

(i) In the problem considered by Gerchberg in [43], the object domain constraint is again $x \cdot 1_{\mathcal{C} D} = 0$ and the image domain constraint is $\mathcal{F} \cdot 1_\Omega = f$, i.e., the Fourier transform of $x$ (not just its modulus) on a domain $\Omega$ is a known function $f$. This constraint forms a convex set (actually an affine subspace, i.e., the translation of a vector subspace). Hence, the resulting feasibility problem is convex (affine), which explains the good convergence properties of the alternating projection algorithm proposed by Gerchberg to solve it. This observation was made by Youla [86] in the case of the Papoulis extrapolation algorithm for band-limited signals [68] (this algorithm is identical to Gerchberg’s, except that the roles played by the object and image domains are interchanged).

(ii) In some problems, e.g., in holography, the image domain constraint arises from the knowledge of the phase $\varphi$ of the Fourier transform of $x$ rather than from its modulus. In [88], Youla observed that the phase constraint $\angle \mathcal{F} = \varphi$ leads to a convex set (actually a convex cone). This fact was fully exploited in [58] (see also [60] and references therein).

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10 If $x \in \mathcal{L}$, then $\Re(x)$ denotes the function defined by $\Re(x) : t \mapsto \Re(x(t))$, and $(\Re(x))^+$ subsequently denotes the positive part: $(\Re(x))^+ : t \mapsto \max\{0, \Re(x(t))\}$.
(iii) In [88], Youla pointed out that the submodulus constraint $|\hat{x}| \leq m$ is convex. In most phase
to retrieval problems, this convexification of the exact constraint is too coarse and it will typically
produce poor results.

(iv) Convexity is an algebraic notion which, by definition, depends on the choice of the underlying
vector space structure. In [18], Çetin exhibited an alternative (discrete) signal space in which
the constraint $|\hat{x}| = m$ is convex. Unfortunately, this approach is not suitable for the phase
retrieval problem since the constraint $x \cdot 1_{c,D} = 0$ is no longer convex in this space.

Ultimately, the difficulty of the phase retrieval problem is caused by the lack of convexity of the
image domain constraint and the lack of good convex approximations to it.

### 3.3 Some fixed point theory

Throughout this section, we continue to make assumption (11).

**Definition 3.17 (firm nonexpansivity and nonexpansivity)** Suppose $T$ is a map from $\mathcal{H}$ to
$\mathcal{H}$. Then $T$ is **firmly nonexpansive**, if

$$
\|T(x) - T(y)\|^2 + \|(I - T)(x) - (I - T)(y)\|^2 \leq \|x - y\|^2, \quad \text{for all } x, y \in \mathcal{H};
$$

and $T$ is **nonexpansive**, if

$$
\|T(x) - T(y)\| \leq \|x - y\|, \quad \text{for all } x, y \in \mathcal{H}.
$$

Fact 3.9 states that the projector onto a nonempty closed convex set is firmly nonexpansive.
From the definition, it is immediate that every firmly nonexpansive map is nonexpansive. Firmly
nonexpansive and nonexpansive maps are actually very closely related:

**Fact 3.18** [45, Theorem 12.1] Suppose $T$ is a map from $\mathcal{H}$ to $\mathcal{H}$. Then the following are equivalent:

(i) $T$ is firmly nonexpansive;
(ii) $2T - I$ is nonexpansive;
(iii) $T = \frac{1}{2} \tilde{T} + \frac{1}{2} I$, for some nonexpansive map $\tilde{T}$.

Many problems can be reduced to finding a fixed point of a nonexpansive mapping: the **fixed
point set** of a mapping $T$ from $\mathcal{H}$ to $\mathcal{H}$ is

$$
\text{Fix } T = \{x \in \mathcal{H} : T(x) = x\}.
$$
For example, the set of fixed points of a projector onto a closed convex set $C$ is just $\text{Fix} P_C = C$.

Fixed points are usually found as limit points of sequences. Discussing convergence in infinite-dimensional spaces requires care, because there exist distinct notions of convergence. The following concepts are appropriate in our present setting:

**Definition 3.19 (norm and weak convergence)** Suppose $(x_n)$ is a sequence in $\mathcal{H}$ and $x \in \mathcal{H}$. Then:

1. $(x_n)$ converges (in norm, or strongly) to $x$, if $\|x_n - x\| \to 0$; in symbols $x_n \to x$.
2. $(x_n)$ converges weakly to $x$, if $\langle x_n - x, y \rangle \to 0$, for all $y \in \mathcal{H}$; in symbols $x_n \rightharpoonup x$.

Physically, $x_n \to x$ means that the energy of the residual signal $\|x_n - x\|^2$ becomes arbitrarily small as $n$ increases; $x_n \rightharpoonup x$ means only that any scalar linear observation of the residual signal becomes arbitrarily small.

It is easy to see that if a sequence converges in norm, then it does so weakly (indeed, if $x_n \to x$ and $y \in \mathcal{H}$ then, by the Cauchy-Schwarz inequality: $|\langle x_n - x, y \rangle| \leq \|x_n - x\| \cdot \|y\| \to 0$); in finite-dimensional spaces, the converse is true as well. However, in every infinite-dimensional space, there exist sequences that converge weakly but not in norm (for instance, every orthonormal sequence converges weakly to zero, but not in norm).

We shall see in the next section that it is very desirable to have algorithms that find fixed points of nonexpansive mappings. The strong interest in firmly nonexpansive mappings stems from the ease of finding their fixed points by simple iteration:

**Fact 3.20 (Opial)** [67, Theorem 3] Suppose $T$ is a firmly nonexpansive mapping from $\mathcal{H}$ to $\mathcal{H}$ with $\text{Fix} T \neq \emptyset$. Then for every $x \in \mathcal{H}$, the sequence $(T^n x)$ converges weakly to some point in $\text{Fix} T$.

**Remark 3.21** An ingenious example by Genel and Lindenstrauss [42] shows that it is not possible to strengthen the conclusion of Fact 3.20 to norm convergence. Also, if $T$ is nonexpansive (but not firmly), then $(T^n x)$ need not converge to a fixed point: consider $T = -I$. Then $\text{Fix} T = \{0\}$ and $T^n x = (-1)^n x$ is not convergent for all $x \neq 0$.

4 Classical algorithms and connections

We discuss three popular algorithms designed for solving the phase retrieval problem (5). Given a starting point $x_0$, every algorithm constructs a sequence $(x_n)$ of functions that in practice often
converges to a solution of (5). The features common to all three algorithms are these: the construction of a function \( x_{n+1} \) depends only on the predecessor \( x_n \), and \( x_{n+1} \) is found by applying the projection operators \( P_S \) and \( P_M \) in some fashion to \( x_n \). Because of this, each algorithm is entirely characterized by its updating rule.

We follow Fienup's framework [39]. To bring out the results as clearly as possible, we set Fienup's parameter \( \beta \) equal to 1 and assume that the object domain constraint is only a support constraint.

### 4.1 Gerchberg-Saxton algorithm

The Gerchberg-Saxton algorithm, also termed the output-output algorithm by Fienup, updates a current iterate \( x_n \) via

\[
x_{n+1}(t) = \begin{cases} (P_M(x_n))(t), & \text{if } t \in D; \\ 0, & \text{otherwise.} \end{cases}
\]

Hence \( x_{n+1} = 1_D \cdot P_M(x_n) \); equivalently, by Example 3.5,

\[
x_{n+1} = (P_S P_M)(x_n).
\]

Local convergence results for this and other types of nonconvex successive projection methods can be found in [28].

### 4.2 Fienup's basic input-output (BIO) algorithm

The update \( x_{n+1} \) in the BIO algorithm is obtained from \( x_n \) by setting

\[
x_{n+1}(t) = \begin{cases} x_n(t), & \text{if } t \in D; \\ x_n(t) - (P_M(x_n))(t), & \text{otherwise.} \end{cases}
\]

Note that \( x_{n+1} = 1_D \cdot x_n + 1_D \cdot (x_n - P_M(x_n)) = x_n - (1 - 1_D) \cdot P_M(x_n) \), which we rewrite as

\[
x_{n+1} = (P_S P_M + I - P_M)(x_n).
\]

### 4.3 Fienup's hybrid input-output (HIO) algorithm

The HIO algorithm constructs the successor of \( x_n \) via

\[
x_{n+1}(t) = \begin{cases} (P_M(x_n))(t), & \text{if } t \in D; \\ x_n(t) - (P_M(x_n))(t), & \text{otherwise.} \end{cases}
\]
Thus
\[ x_{n+1} = 1_D \cdot P_M(x_n) + 1_D \cdot (x_n - P_M(x_n)) \]
\[ = 1_D \cdot P_M(x_n) + (1 - 1_D) \cdot (x_n - P_M(x_n)) \]
\[ = 1_D \cdot (2P_M(x_n) - x_n) + x_n - P_M(x_n) \]
\[ = (P_S(2P_M - I) + (I - P_M))(x_n). \] (22)

Note that this can also be written as
\[ x_{n+1} = (P_S P_M + (I - P_S)(I - P_M))(x_n), \] (23)
because the projector onto a closed linear subspace is linear.

**Remark 4.1** The description of the HIO algorithm in [39], specialized to our setting, actually reads\(^{11}\)
\[ x_{n+1}(t) = \begin{cases} 
(P_M(x_n))(t), & \text{if } t \in D \text{ or } (P_M(x_n))(t) = 0; \\
(x_n(t) - (P_M(x_n))(t)), & \text{otherwise}. 
\end{cases} \] (24)

The updates differ precisely at points \( t \) that belong to \( CD \) and that satisfy \( (P_M(x_n))(t) = 0 \). However, it is not easy to determine which formulation is used in the community, as the papers we are aware of are not specific on this question. Notable exceptions are [79, 78, 77], which use the formulation we shall employ\(^{12}\), and [40], which uses the literal definition. We should note that the main results of this paper would remain essentially unchanged if we followed the literal definition (see Footnote 18). Interestingly, the same issue formally arises for the output-output algorithm and for the BFO algorithm; however, a careful inspection reveals that this does not lead to different algorithms!

### 4.4 Main observations

We are now ready to establish the formal correspondence between classical algorithms for solving (5) and their counterparts for solving a two-set convex feasibility problem.

\(^{11}\)The reason for this difference is that Fienup defines on page 2763 of [39] \( \gamma \) as the set of all points where (in our notation) \( P_M(x_n) \) violates the object domain constraints. Hence \( \gamma = \{ t \in CD : (P_M(x_n))(t) \neq 0 \} \), or: \( t \in \gamma \) if and only if \( t \not\in D \) and \( P_M(x_n)(t) \neq 0 \). It follows that \( t \) belongs to the complement of \( \gamma \) if and only if \( t \in D \) or \( P_M(x_n)(t) = 0 \). The latter condition then leads to this different interpretation of the HIO algorithm. Sticking with this interpretation for another moment, we could set \( D(n) = D \cup \{ CD : P_M(x_n)(t) = 0 \} \), \( S(n) = \{ z \in L : z \cdot 1_C \neq 0 \} \), and obtain analogously
\[ x_{n+1} = (P_S(n)(2P_M - I) + (I - P_M))(x_n). \]

\(^{12}\)The corresponding mask is certainly much easier to implement.
4.4.1 The convex feasibility problem

Assume $A$ and $B$ are two closed convex sets in a real Hilbert space $\mathcal{H}$. The associated convex feasibility problem is to

\[
\text{find } x \in A \cap B.
\]

Note the similarity between (25) and the formulation (5) of the phase retrieval problem as a feasibility problem. However, (5) is **not** a convex feasibility problem as the image modulus constraint is not convex (Example 3.16).

We now revisit the three classical algorithms for solving the phase retrieval problem described above. It will turn out that each algorithm corresponds to a classical algorithm for solving (25).\(^{13}\)

4.4.2 Gerchberg-Saxton algorithm and POCS

The method of alternating projections onto convex sets (POCS) generates, for the present setting of two constraints, sequences $(a_n)$ and $(b_n)$ as follows: pick an arbitrary starting point $a_0 \in \mathcal{H}$. Then update by

\[
b_n = P_B(a_n) \quad \text{and} \quad a_{n+1} = P_A(b_n), \quad \text{for all } n \geq 0. \tag{26}
\]

This process is depicted in Fig. 1.

The following basic result shows that POCS does find a solution of (25):

**Fact 4.2 (Bregman)** [14] Suppose $A \cap B \neq \emptyset$. Then both sequences $(a_n)$ and $(b_n)$ converge weakly to a point in $A \cap B$.

**Remark 4.3** Although we do not go into details, we mention in passing two possible paths to proving Fact 4.2 The first approach is fixed point theoretic and consists of showing that (i) $\text{Fix}(P_A P_B) = A \cap B$ and (ii) iterating the composition $P_A P_B$ produces — analogously to Fact 3.20 — fixed points\(^{14}\). The second approach is more elementary and builds on the notion of Fejér monotonicity; see [10, 7, 25].

**Observation 4.4 (Gerchberg-Saxton algorithm as POCS)** Identify $A$ with the object domain constraint set $S$, and $B$ with the image domain constraint set $M$. Then the sequence $(a_n) = ((P_A P_B)^n(a_0))$ corresponds to the sequence $(x_n)$ generated by the Gerchberg-Saxton algorithm (Section 4.1). This connection was established by Levi and Stark [59] in 1984.

\(^{13}\)The algorithms discussed here for solving (25) can be viewed in the broader context of finding a zero of the sum of two maximal monotone operators. Good starting points are [24, 34, 35].

\(^{14}\)Unfortunately, $P_A P_B$ is generally not firmly nonexpansive. However, it is strongly nonexpansive, and, for this class of mappings, a result corresponding to Fact 3.20 does exist. See [15] for further information.
4.4.3 Fienup's BIO algorithm and Dykstra's algorithm

Dykstra's algorithm was first developed for closed convex cones in [33], and subsequently generalized to closed convex sets in [13]. For two closed convex sets $A$ and $B$, it produces four sequences $(a_n)$, $(b_n)$, $(p_n)$, and $(q_n)$ as follows (see Fig. 2). Fix a starting point $a_0$, set $q_0 = 0 = p_0$, and update for $n \geq 0$ via

$$
\begin{align*}
  b_n &= P_B(a_n + q_{n-1}), \quad q_n = (I - P_B)(a_n + q_{n-1}) = a_n + q_{n-1} - b_n, \\
  a_{n+1} &= P_A(b_n + p_n), \quad p_{n+1} = (I - P_A)(b_n + p_n) = b_n + p_n - a_{n+1}.
\end{align*}
$$

(27)

Clearly, Dykstra's algorithm is more involved than POCS and is more demanding in terms of storage; however, its convergence properties are superior.

\textbf{Fact 4.5 (Boyle-Dykstra) [13]} Suppose $A \cap B \neq \emptyset$. Then both sequences $(a_n)$ and $(b_n)$ converge in norm to $P_{A \cap B}(a_0)$, the point in $A \cap B$ closest to $a_0$.

Fact 4.5 is quite remarkable because the sequences converge in norm, and their limit is explicitly identified as the nearest feasible point to the starting point. This explains the popularity of Dykstra's algorithm in approximation theory, where this method is well understood and many extensions have been found; see, for instance, [9, 12, 41, 48, 53]. For applications of Dykstra's algorithm to signal recovery, see [21].

For the rest of this subsection, we assume additionally that $A$ is a \textit{closed linear subspace}. Then $(p_n)$ lies entirely in $A^\perp$, the orthogonal complement of $A$, and the computation of $a_{n+1}$ becomes $a_{n+1} = P_A b_n + P_A p_n = P_A b_n$. Thus, the sequence $(p_n)$ is not needed, and Dykstra's algorithm simplifies to:

$$
\begin{align*}
  b_n &= P_B(a_n + q_{n-1}), \quad q_n = (I - P_B)(a_n + q_{n-1}), \quad a_{n+1} = P_A(b_n).
\end{align*}
$$

(28)

Hence

$$
a_{n+1} + q_n = (P_A P_B + I - P_B)(a_n + q_{n-1}) = (P_A P_B + I - P_B)^{n+1}(a_0).
$$

(29)

The next observation appears to be new.

\textbf{Observation 4.6 (BIO algorithm as Dykstra's algorithm)} As before, let us identify $A$ with the object domain constraint set $S$, and $B$ with image domain constraint set $M$. Then the sequence $(a_n + q_{n-1})$ corresponds precisely to the sequence $(x_n)$ generated by Fienup's BIO algorithm (see Section 4.2).

\footnote{In the aforementioned context of maximal monotone operators (Footnote 13), Dykstra's algorithm can be interpreted as a tight version of the Peaceman-Rachford algorithm. See [34, page 77] for further information. Let us also note that in the standard linear case, the Peaceman-Rachford and Douglas-Rachford algorithms can be viewed from a unifying standpoint [83, Section 7.4].}
Remark 4.7 Even when $A \cap B \neq \emptyset$, it is possible that the sequences $(p_n)$ and $(q_n)$ generated by Dykstra’s algorithm (in its general form) are both unbounded; see [48]. This suggests the pertinent sequence to monitor in Fienup’s BIO algorithm is $(P_M(x_n))$, rather than $(x_n)$.

4.4.4 Fienup’s BIO algorithm and the Douglas-Rachford algorithm

When specialized to the convex feasibility problem (25), the Douglas-Rachford algorithm\(^{16}\) generates a sequence $(x_n)$, from an arbitrary starting point $x_0$, by

$$x_{n+1} = (P_A(2P_B - I) + (I - P_B))(x_n).$$  \hspace{1cm} (30)

For brevity, we set

$$T = P_A(2P_B - I) + (I - P_B).$$  \hspace{1cm} (31)

If $A$ is a closed linear subspace, then $T$ can be written more symmetrically as $T = P_A P_B + (I - P_A)(I - P_B)$. Now let $R_A = 2P_A - I$ be the reflector with respect to $A$ and define $R_B$ likewise. The following proposition gives an alternative description of the Douglas-Rachford algorithm that lends itself to a simple geometrical interpretation (see Fig. 3).

Proposition 4.8 The mapping $T$ in (31) can be written as $T = (R_A R_B + I)/2$. Hence, (30) is equivalent to

$$x_{n+1} = \frac{1}{2}(R_A R_B + I)(x_n).$$  \hspace{1cm} (32)

The next two basic results on the Douglas-Rachford iteration are due to Lions and Mercier [61]; see also [35, 34]. We include some proofs in Appendix A, as they appear to be simpler than those found in the literature.

Fact 4.9 The mapping $T$ in (31) is firmly nonexpansive.

Fact 4.10 (Lions-Mercier) Suppose $A \cap B \neq \emptyset$. Then the sequence $(x_n)$ converges weakly to some point $x \in \text{Fix} T$ and $P_B(x) \in A \cap B$. Moreover, the sequence $(P_B(x_n))$ is bounded, and every weak cluster point\(^{17}\) of $(P_B(x_n))$ lies in $A \cap B$. If $\mathcal{H}$ is finite-dimensional, then $x_n \to x$ and $P_B(x_n) \to P_B(x) \in A \cap B$.

The following connection does not seem to have been drawn elsewhere.

\(^{16}\)The Douglas-Rachford algorithm was originally developed as a linear implicit iterative method to solve partial differential equations in [32] (see also [83, Chap. 7]). It was extended to an operator splitting method for finding a zero of the sum of two maximal monotone operators by Lions and Mercier in [61]. Applied to the normal cone maps of the constraints sets, one obtains a method for solving (25). See [24, 34, 35, 61] for further information.

\(^{17}\)u is a weak cluster point of a sequence $(u_n)$ if there exists a subsequence $(u_{n_k})$ such that $u_{n_k} \rightharpoonup u$. 

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Observation 4.11 (HIO algorithm as Douglas-Rachford algorithm) Let us identify $A$ with the object domain constraint $S$, and $B$ with the image domain constraint $M$. Then the sequence generated by the HIO algorithm (see Section 4.3) corresponds exactly to the sequence generated by the Douglas-Rachford algorithm\textsuperscript{18}.

5 Concluding remarks

The contribution of this paper is two-fold. First, a formal numerical analysis of the phase retrieval problem has been carried out in the mathematical context of multi-valued projection operators. This analysis provides rigorous and easily verifiable criteria for calculating projections. Second, new connections have been established between some classical phase retrieval methods and some standard convex optimization algorithms.

While the mathematical theory remains unable to completely analyze the convergence behavior of these algorithms in nonconvex settings, the analogies drawn here open the door for experimentation with variations that are well understood in convex settings. We believe that the convex-analytical viewpoint adopted in this paper can be exploited further in order to develop alternative phase retrieval schemes.

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\textsuperscript{18}If we had used the literal update rule for the HIO algorithm, the present observation would change only in one aspect: the constraint $A$ from (25) would be identified with $S(n)$ (see Remark 4.1 and Footnote 11), and hence vary with $n$. 

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Appendix A – Proofs

Proof of Example 3.5: See Example 3.14 for a rigorous proof. □

Proof of Example 3.6: This is a sketch of the proof; see [63, Theorem 4.2] for full details. Fix \( x \in L \). Because the Fourier transform is unitary, it follows that \( \hat{P}_M(x) = P_A(\hat{x}) \). In turn, by Theorem 3.15, the projection onto \( A \) can be found separably, provided that the selection is measurable. But the projection onto the circle in \( \mathbb{C} \) is easy: radially scale the point, and observe the multi-valuedness at the origin. The step from the measurable selection to the measurable phase \( \varphi \) requires a measure-theoretical argument; see [63, Proof of Theorem 4.2]. □

Proof of Fact 3.9: This is part of the folklore. See, for instance, [89, Lemma 1.1]. □

Proof of Proposition 3.11: (i) Easily verified. (ii) Let \( (a_n) \) be a sequence in \( A \) converging to some \( z \in L \). A result by Riesz (see [1, Theorem 12.6] or [30, Theorem 2.8.2]) implies that there exists a subsequence \( (a_{k_n}) \) of \( (a_n) \) that converges to \( z \) a.e. Since \( (a_{k_n}(t)) \) lies in the closed set \( A(t) \) a.e., it follows that \( z(t) \in A(t) \) a.e. Consequently, \( A \) is closed. □

Proof of Theorem 3.12: Since each set \( A(t) \) is closed convex and nonempty, the function \( y : t \mapsto P_{A(t)}(x(t)) \), is well-defined. By [2, Corollary 8.2.13(3)] (see also [69, Exercise 14.17(b)]), \( y \) is measurable. Pick an arbitrary \( a \in A \). Then \( |x(t) - y(t)| \leq |x(t) - a(t)| \) a.e. Squaring and integrating yields \( y \in L \) and hence \( y \in P_A(x) \). Since \( A \) is closed and convex by Proposition 3.11, \( P_A(x) \) is a singleton (Fact 3.9) in which case one writes \( y = P_A(x) \), by convention. □

Proof of Example 3.14: We first define \( A(t) \), depending on the case considered: (i) \( A(t) = \mathbb{C} \), if \( t \in D \), \( A(t) = \{0\} \), otherwise, (ii) \( A(t) = \mathbb{R} \), (iii) \( A(t) = \mathbb{R}_+ \), respectively. Now let \( A \) be as in Theorem 3.12. Note that \( A \) is nonempty, as it contains the zero function. Next, observe that in each case, we can write \( A(t) = \mu(t) \cdot Z \), where \( \mu : \mathbb{R}^N \to \mathbb{C} \), is measurable and \( Z \subseteq \mathbb{C} \): indeed, (i) \( Z = \mathbb{C} \) and \( \mu = 1_D \), (ii) \( Z = \mathbb{R} \) and \( \mu \equiv 1 \), (iii) \( Z = \mathbb{R}_+ \) and \( \mu \equiv 1 \). In view of Remark 3.13, \( t \mapsto A(t) \) is a measurable multifunction in each case. The result now follows from Theorem 3.12. □

Proof of Theorem 3.15: Since \( A \) is a measurable multifunction, the function \( \mathbb{R}^N \to \mathbb{R}_+ : t \mapsto d(\rho, A(t)) \) is measurable, for every \( \rho \in \mathbb{C} \) [2, Corollary 8.2.13(2)], [69, Theorem 14.3]. On the other hand, the function \( \mathbb{R}^N \to \mathbb{R}_+ : \rho \mapsto d(\rho, A(t)) \) is continuous (even nonexpansive), for every \( t \in \mathbb{R}^N \). Altogether, the function \( \mathbb{R}^N \times \mathbb{C} \to \mathbb{R}_+ : (t, \rho) \mapsto d(\rho, A(t)) \) is a Carathéodory function; see [1, Definition 4.49]. Now \( A \neq \emptyset \), hence \( A \) is nonempty-valued. By [1, Theorem 17.5], the multifunction \( A \) is weakly measurable. Now fix \( \rho \in \mathcal{L} \) and let \( f : \mathbb{R}^N \times \mathbb{C} \to \mathbb{R}_+ : (t, \rho) \mapsto |x(t) - \rho| \). Then \( f(t, \rho) \) is measurable in \( t \), and continuous in \( \rho \); thus, \( f \) is also a Carathéodory function. The Measurable Maximum Theorem [1, Theorem 17.18] yields (i) \( t \mapsto d(x(t), A(t)) \) is a measurable function, (ii) \( t \mapsto P_{A(t)}(x(t)) \) is a measurable multifunction, with nonempty compact values, and (iii) there exists a measurable selection \( z(t) \in P_{A(t)}(x(t)) \). Clearly, \( z \in P_A(x) \). (This first part of the proof can also be contemplated from a higher perspective, see [2, Section 8.2]). It remains to prove the equivalence concerning \( P_A(x) \). The “if” part is clear. We now suppose to the contrary that the “only if” part is false. Then there exists a measurable function \( y \in P_A(x) \) such that the
set \( E = \{ t \in \mathbb{R}^N : |x(t) - y(t)| > d(x(t), A(t)) \} \) has strictly positive measure. But then
\[
d^2(x, A) = \| x - y \|^2
\]
\[
= \int_E |x(t) - y(t)|^2 dt + \int_{E'} |x(t) - y(t)|^2 dt
\]
\[
> \int_E |x(t) - z(t)|^2 dt + \int_{E'} |x(t) - z(t)|^2 dt
\]
\[
= \| x - z \|^2 = d^2(x, A),
\]
which is absurd. \( \square \)

**Proof of Example 3.16:** Recall that \( m \in \mathcal{L} \) is the prescribed nonnegative modulus function. The set of all functions satisfying the image modulus constraint is \( M = \{ z \in \mathcal{L} : |z| = m \} \). Note that \( M \neq \emptyset \), since \( m \in M \). Let \( S \) be the closed unit circle in \( \mathbb{C} \), and set \( A(\omega) = m(\omega) \cdot S \), for all \( \omega \in \mathbb{R}^N \). The multifunction \( \omega \mapsto A(\omega) \) is compact-valued, nonempty-valued, and measurable (Remark 3.13). Now let \( A = \{ z \in \mathcal{L} : z(\omega) \in A(\omega) \ a.e. \} \). Then \( A \) is closed by Proposition 3.11.(i). Consequently, as the pre-image of \( A \) under a continuous operator, namely the the Fourier transform, the set \( M \) is closed\(^{19} \). \( \square \)

**Proof of Proposition 4.8:** By simple expansion
\[
R_A R_B = (2P_A - I)(2P_B - I) = 2P_A(2P_B - I) + 2(I - P_B) - I.
\]
Hence, (31) yields \( T = \frac{1}{2}(R_A R_B + I) \). \( \square \)

**Proof of Fact 4.9:** (See also [61, Proposition 2],[35, Corollary 4.1].) Because \( P_A \) and \( P_B \) are projectors onto convex sets, they are firmly nonexpansive (Fact 3.9). Thus, by Fact 3.18, \( R_A \) and \( R_B \) are nonexpansive. Hence the composition \( R_A R_B \) is nonexpansive which, in turn, implies (Fact 3.18 again) that \( \frac{1}{2}(R_A R_B + I) \) is firmly nonexpansive. In view of Proposition 4.8, the proof is complete. \( \square \)

**Fact A1** \( P_B(\text{Fix } T) = A \cap B \subseteq \text{Fix } T \).

**Proof.** Fix an arbitrary \( x \in \mathcal{H} \). Write \( x = b + q \), where \( b = P_B(x) \) and \( q = x - b \). Then \( b = P_B(x) = P_B(b + q) \). The result follows from the equivalences \( x = T(x) \Leftrightarrow x = P_A(2P_B - I)(x) + (I - P_B)(x) \Rightarrow b + q = P_A(2b - (b + q)) + b + q - b \Leftrightarrow b = P_A(b - q) \). \( \square \)

**Proof of Fact 4.10:** (See also [61, Theorem 1.(iii) and Remark 7] and [35, Corollary 6.1]\(^{20} \)) By Fact A1, \( \text{Fix } T \) contains \( A \cap B \neq \emptyset \). In view of Fact 3.20, the sequence \( (x_n) = (T^n(x_0)) \) converges

\(^{19}\)However, as shown in [63, Property 4.1], the set \( M \) is not weakly closed, i.e., if a sequence \( (x_n) \) of points in \( M \) converges weakly to a point \( x \), then \( x \) may not be in \( M \).

\(^{20}\)While \( P_B \) is nonexpansive and therefore Lipschitz continuous, this property is not sufficient to draw the conclusion advertised in [35, Corollary 6.1], namely (in our context), that \( (P_B x_n) \) converges weakly to a point in \( A \cap B \). Such a conclusion requires additional assumptions, e.g., that \( P_B \) be weakly continuous (if so, then \( P_B x_n \), \( \xrightarrow{w} P_B x \)), as is the case when \( \dim \mathcal{H} < +\infty \). Note, however, that the projector onto a closed convex set may fail to be weakly continuous [89, Example on page 245].
weakly to some fixed point \( x \) of \( T \). Since \( P_B \) is nonexpansive and \( (x_n) \) is bounded, it follows that 
\( (P_B(x_n)) \) is bounded. Since \( T \) is firmly nonexpansive, 
\[
\|x_n - x\|^2 \geq \|x_{n+1} - x\|^2 + \|x_n - x_{n+1}\|^2,
\]
which implies (after telescoping)
\[
0 \leftarrow x_n - x_{n+1} = P_B(x_n) - P_A(2P_B(x_n) - x_n). \tag{A3}
\]
Hence weak cluster points of \( (P_B(x_n)) \) must lie in \( A \cap B \). Finally, if \( \mathcal{H} \) is finite-dimensional, then 
\( P_B(x_n) \to P_B(x) = P_A(2P_B(x) - x) \in A \). \( \square \)

References


Figure 1: POCS algorithm. The initial point $a_0$ is projected onto $B$ and then onto $A$. The point $a_1$ thus obtained belongs to both sets and the algorithm therefore converges in two steps. Note that the solution $a_1$ is not the projection of $a_0$ onto $A \cap B$. 
Figure 2: Dykstra’s algorithm. The first two steps of this algorithm are identical to those of POCS (Fig. 1). Here, however, although $a_1 \in A \cap B$, the algorithm does not reach convergence at this point since the outward pointing normal $q_0$ pulls the vector $a_1 + q_0$ out of $B$ before it is projected onto $B$. Through this process, two infinite sequences $(a_n)$ and $(b_n)$ are generated that converge to $P_{A \cap B}(a_0)$. Note that since $A$ is an affine subspace in this example, $p_n \perp A$ and, therefore, $a_{n+1} = P_A(b_n)$. 

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Figure 3: Douglas-Rachford algorithm. The update equation (32) is executed as follows: one first computes the reflection $r_{n+\frac{1}{2}}$ of $x_n$ with respect to $B$ and then the reflection $r_{n+1}$ of $r_{n+\frac{1}{2}}$ with respect to $A$. The update $x_{n+1}$ is the midpoint of the segment between $x_n$ and $r_{n+1}$. In this example the algorithm converges in 4 iterations since $x_4 \in A \cap B$. 