A $C^1$-FUNCTION THAT IS EVEN ON A SPHERE AND HAS NO CRITICAL POINTS IN THE BALL

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ABSTRACT. In this article we construct a real-valued $C^1$-function on the closed ball in $\mathbb{R}^2$ that is even on the boundary of the ball, and has no critical points inside the ball. This provides a counterexample to a nonsmooth Rolle-type theorem sought in [BF].

1. INTRODUCTION

In [BF] Borwein and Fitzpatrick use Schauder's fixed point theorem to obtain various duality inequalities similar or equivalent to the nonsmooth mean-value inequalities introduced in [CL] and [LR]. In particular, two striking corollaries are proved (Corollaries 3.6 and 3.7 of [BF]):

**Theorem 1.1.** Let $B$ be the closed unit ball in $\mathbb{R}^n$ and let $g : B \to \mathbb{R}$ be Lipschitz. Then there is $x^* \in \partial g(B)$ such that

$$||x^*||_\ast \leq \max_{a \in B} (g(a) - g(-a))/2.$$ 

**Theorem 1.2.** Let $B$ be the closed unit ball in $\mathbb{R}^n$ and let $g : B \to \mathbb{R}$ be Lipschitz. Then there is $x^* \in \partial g(B)$ such that

$$||x^*||_\ast \leq \max_{a \in B} g(a) - g(0).$$

In the above $\partial g(x)$ denotes the Clarke subdifferential of $g$ at $x$.

Using a variational argument, the maximum in Theorem 1.2 can be restricted to the boundary of the ball, yielding the following inexact 'maximum principle' (see Corollary 3.9):

**Theorem 1.3.** Let $B$ be the closed unit ball in a Banach space $X$ and let $g : B \to \mathbb{R}$ be Lipschitz. Then

$$\inf \{||x^*||_\ast : x^* \in \partial g(x), ||x|| \leq 1\} \leq \sup_{a \in \partial B} |g(a)|.$$ 

If $g$ vanishes on the sphere in finite dimensions this becomes a version of Rolle's theorem asserting that a function vanishing on the sphere has a critical point interior to the ball. One might ask whether a similar improvement might be made to Theorem 1.1: must there be $x \in \partial g(B)$ such that

$$||x^*||_\ast \leq \max_{a \in \partial B} (g(a) - g(-a))/2 ?$$

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This article provides a very strong negative answer: (1.1) need not hold, even when the function considered is $C^1$ and the domain is $\mathbb{R}^2$. To this end, we construct an example of a $C^1$-function $G : \mathbb{R}^2 \to \mathbb{R}$ such that $G|_S$ is an even function, so that
$$\max_{a \in \partial B} (g(a) - g(-a))/2 = 0,$$
while $0 \notin \nabla G(B) = \partial G(B)$.

Note that (1.1) would be a strong (nonsmooth) analogue of Rolle’s theorem. A related counter-example can be found in [F] showing that a smooth exact version of Rolle’s theorem fails in $l_2$. There is a large literature on the failure of Rolle’s theorem in infinite dimensions; a representative recent paper is [AGJ]. The existence of our counter-example suggests that a direct variational proof of Theorem 1.1 is unlikely to be found.

2. Notation

Throughout $\|\cdot\|$ refers to the 2-norm in $\mathbb{R}^2$. We denote by $B$ the closed unit 2-ball centered at 0, by $O$ the interior of $B$, and $S = B \setminus O$. For $\phi \in \mathbb{R}$, let $s(\phi) \in S$ denote the point of argument $\phi$. Whenever $x \in \mathbb{R}^2$ is expressed as $x = (\cdot, \cdot)$, we mean its polar coordinates, whereas $x = (\cdot, \cdot)$ will be used for the Cartesian ones. For $x, y \in \mathbb{R}^2$, $[x, y]$ denotes the closed line-segment with endpoints $x$ and $y$. Finally, $I_\Delta$ denotes the indicator function of the interval $\Delta$:

$$I_\Delta(x) := \begin{cases} 1 & x \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

3. Construction

The desired function $G$ is constructed in two stages: first we define a function $G_0$ with no critical points on the sphere, and then we apply lemma 3.1 to $G_0$ in order to even it out on $S$ without introducing any new critical points on $B$. The construction is quite technical and so we accompany all steps with pictures.

3.1. Defining Calderas. We start by defining a function $F_0$ that generates a caldera-shaped graph, and we center translates of $F_0$ at three different points in the plane; at $A = (\sqrt{2}, \pi/4)$, $B = (\sqrt{2}, \pi)$, and $C = (\sqrt{2}, 3\pi/2)$ (Figure 2). $F_0$ is defined as follows.

Let $f : [0, +\infty) \to \mathbb{R}$ be defined by

$$f(\rho) := \begin{cases} \cos^2(\frac{\pi}{6}(\rho - 1)) & \rho \in [0, 1.6] \\ 0 & \rho \in [0, 0.4] \cup [1.6, +\infty) \end{cases}$$

Note that:

1. $f \in C^1([0, +\infty))$;
2. $f'|_{[0,1,1.6]} > 0$;
3. $f'|_{[1,1,1.6]} < 0$;
4. $f(1 + \rho) = f(1 - \rho)$ for $\rho \in [0, 1]$.

Given some $\theta \in \mathbb{R}$, define $F_0 \in C^1(\mathbb{R}^2)$ by

$$F_0((\sqrt{2}, \theta) + (\rho, \phi)) := f(\rho);$$

This defines a caldera centered at $(\sqrt{2}, \theta)$ (Figure 1). Let

$$F := F_0 - F_0 - F_0;$$

this defines one upright caldera centered at $A$ and two inverted calderas at $B$ and $C$ (Figure 3). Since calderas $B$ and $C$ are inverted, the critical points of $F$ in the first quadrant are exactly the critical points of $F_0$, i.e. the rim of caldera $A$. 

In the second quadrant, the critical points of $F$ are a subset of the critical points of $F_\pi$, since caldera $\text{C}$ removes those critical points of $F_\pi$ that are within the support of $F_{\frac{3\pi}{2}}$. By symmetry, the critical points of $F$ in the fourth quadrant are exactly those critical points of $F_{\frac{3\pi}{2}}$ not in the support of $F_\pi$.

In the third quadrant, we have a critical point only at $(1, \frac{5\pi}{4})$.

### 3.2. Removing Critical Points

Each of the four quadrants contain critical points that we wish to remove. In the first quadrant, we add a thin rising ridge to the top of caldera $\text{A}$. In the second and fourth quadrants, we add a narrow gorge to the bottom of calderas $\text{B}$ and $\text{C}$. The following function $f_0$ will serve as the radial component of these topographical features.

![Figure 1. $F_\theta$.](image1)

![Figure 2. The three calderas are centered at $\text{A}$, $\text{B}$ and $\text{C}$, and the rim of each caldera is indicated with solid arcs. The support of each caldera is delimited by two dashed arcs.](image2)
Define \( f_0 : [0, +\infty) \to \mathbb{R} \) by
\[
f_0(\rho) := \begin{cases} 
\cos^2(50\pi(\rho - 1)) & \rho \in [0.99, 1.01] \\
0 & \rho \in [0, 0.99] \cup [1.01, +\infty)
\end{cases}
\]

Then we have:
1. \( f_0 \in C^1([0, +\infty)) \),
2. \( f_0'(\rho) > 0 \) for \( \rho \in (0.99, 1) \),
3. \( f_0'(\rho) < 0 \) for \( \rho \in (1, 1.01) \),
4. \( f_0(1 + \rho) = f(1 - \rho) \) for \( \rho \in [0, 1] \).

In the second quadrant, there is a flat section on a region of \( B + S \). We add a narrow circular valley to the rim of caldera \( B \) to remove these critical points. The angular component of this valley is defined by the following function.

Let \( p_1 \in C^1(\mathbb{R}) \) be a 2\( \pi \)-periodic function such that:
1. \( p_1|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \geq \frac{1}{2} \),
2. \( p_1|_{[-\frac{\pi}{2}, 0]} = 1 \),
3. \( \mu_1[0, \pi] < 0 \),
4. \( p_1(\phi) = \frac{2 - \pi}{4} - \phi \) for \( \phi \in \left[ \frac{\pi}{4}, 0.01, \frac{\pi}{4} \right] \)

(see Figure 4). Define \( G_1 \in C^1(\mathbb{R}^2) \) by

\[
G_1(B + (\rho, \phi)) := -f_0(\rho)p_1(\phi).
\]

This determines a gorge on the bottom of caldera \( B \), which slopes down from a height of \(-\frac{1}{2}\) at \( B + (1, \frac{\pi}{4}) \) to a height of \(-1\) at \( B + (1, -\frac{\pi}{4}) \) (Figure 5). Note that we only care about the values of \( p_1 \) on the interval \([-\frac{\pi}{4}, \frac{3\pi}{4}]\), since this is the angular range of the region \((B + S) \cap B\). In the fourth quadrant, we define a gorge on the flat spots of caldera \( C \) by taking the mirror image of \( G_1 \) along the \( y = x \) line of the plane. Let \( p_2(\phi) := p_1 \left( \frac{\pi}{2} - \phi \right) \) and define \( G_2 \in C^1(\mathbb{R}^2) \) by

\[
G_2(C + (\rho, \phi)) := -f_0(\rho)p_2(\phi).
\]

By symmetry, this defines a gorge on the bottom of caldera \( C \) that takes a value of \(-\frac{1}{2}\) at \( C + (1, \frac{\pi}{4}) \) and decreases to \(-1\) at \( C + (1, \frac{3\pi}{4}) \).

The first quadrant has critical points along the length of the rim of caldera \( A \), that is, on \((A + S) \cap B\). To remove these points, we put a ridge on \( A + S \) rising upward along the arc from \( A + (1, \pi) \) to \( A + (1, \frac{3\pi}{2}) \). Let \( p_3 \in C^1(\mathbb{R}) \) be a \( 2\pi \)-periodic function such that

1. \( p_3[\pi, \frac{3\pi}{2}] \geq \frac{1}{2} \),
2. \( p_3'[\pi, \frac{3\pi}{2}] > 0 \),
3. \( p_3(\phi) = \frac{1}{2} - \pi + \phi \) for \( \phi \in [\pi, \pi + 0.01] \)

(see Figure 6).

We create the desired ridge with the function \( G_3 \in C^1(\mathbb{R}^2) \) defined by

\[
G_3(A + (\rho, \phi)) := f_0(\rho)p_3(\phi) \quad \text{(Figure 8)}.
\]

In the third quadrant, we must remove the critical point at \((1, \frac{3\pi}{2})\). To accomplish this we add a function \( G_4 \) that has positive slope at \((1, \frac{3\pi}{2})\) in the direction \((1, \frac{3\pi}{2})\). The definition of \( G_4 \) is as follows.

Let \( p_4 \in C^1(\mathbb{R}) \) be a \( 2\pi \)-periodic function such that \( p_4[0, \pi] \geq 0 \), \( p_4'[0, \frac{\pi}{2}] < 0 \), and \( p_4'[\frac{\pi}{2}, \frac{3\pi}{2}] = 0 \). We use \( p_4 \) to generate the angular component of \( G_4 \) (Figure 6).
Figure 6. $p_3$.

Figure 7. $p_4$.

Figure 8. $G_3$. 
We define $G_4 \in C^1(\mathbb{R}^2)$ piecewise on a partition of $B$. On $(B + B) \cap B$, $G_4$ is defined by

$$G_4(B + (\rho, \phi)) := -f_0(\rho)p_4 \left( \phi + \frac{\pi}{4} \right) \text{ for } \rho \leq 1.$$ 

On $(C + B) \cap B$, $G_4$ is defined by

$$G_4(C + (\rho, \phi)) := -f_0(\rho)p_4 \left( \frac{3\pi}{4} - \phi \right) \text{ for } \rho \leq 1.$$ 

Finally, for $(x, y) \notin \{(B, C) + B\}$, we define $G_4$ by

$$(3.1) \quad G_4((x, y)) := -f_0(0)p_4 \left( 2 \text{ arctan} \frac{(x\sqrt{2} + 1)^2 + (y\sqrt{2} + 1)^2}{2(x + y + \sqrt{2})} \right).$$

If $C$ is a circle tangent to the line segment $[B, C]$ at the midpoint $M = \langle -\sqrt{2}/2, -\sqrt{2}/2 \rangle$ of $B$ and $C$, then (3.1) makes $G_4$ constant on the arc $D$ defined by the intersection of $C$ with $B \setminus \{(B, C) + O\}$ (Figure 9). To see why this is so, consider Figure 10. For any point $P = \langle x, y \rangle \in D$, there is a unique circle $C$ as above. The center of this circle, $Z = \langle z, z \rangle$, satisfies $||Z - M|| = ||Z - P||$, which means that

$$z = \frac{1}{2} \left( \frac{x^2 + y^2 - 1}{x + y + \sqrt{2}} \right).$$

We wish for $G_4$ to be constant on $D$, and we require that $G_4(D) = G_4(P')$ where $P'$ is the point of intersection of $D$ and $B + S$. Since $\triangle BMZ$ is right angled, the angle $\phi$ satisfies

$$(3.2) \quad \phi = \text{ arctan} \frac{||Z - M||}{||M - B||}$$

$$(3.3) \quad \phi = \text{ arctan} \frac{(x\sqrt{2} + 1)^2 + (y\sqrt{2} + 1)^2}{2(x + y + \sqrt{2})\sqrt{2}}.$$ 

Since the angle $\angle MBP'$ is double the angle $\phi$, the argument of $p_4$ in (3.1) is the same as that of $P'$. The moduli also coincide, since the modulus of $P'$ is 1. Of course, the definition of $G_4$ guarantees that $G_4(P') = G(P'')$, where $P''$ is the point of intersection of $D$ and $C + S$.

We have shown that $G_4$ is continuous; it remains to show that it is continuously differentiable. To see that $G_4$ remains a $C^1$ function after the gluing of the pieces, we observe that:

- for each $x = B + (1, \phi)$, the directional derivative in direction $(1, \phi)$ is zero on both sides and is continuous there, while that in direction $(1, \phi + \frac{\pi}{2})$ equals $f_0(1)p_4'(\phi + \frac{\pi}{2})$ on both sides and is continuous there;
- for each $x = C + (1, \phi)$, a symmetric argument applies.

The definition also guarantees that supp $G_4$ is contained in the third quadrant.

Now let $G_0 \in C^1(\mathbb{R}^2)$ be defined by

$$G_0 := F + G_1 + G_2 + G_3 + G_4.$$ 

To summarize, $G_0$ is composed of the sum $F$ of the three calderas, with the gorges/ridges $G_1, G_2$ and $G_3$ added to remove the critical points on the rims of the three calderas, and the bump $G_4$ added to remove the critical point at $(1, \frac{5\pi}{4})$.
3.3. Verifying that $G_4$ has no Critical Points. We begin at caldera $A$. Note that $G_0(x) = F_\pi(x) + G_3(x)$ whenever $x = A + (\rho, \phi)$ for any $\rho \in [0, 1.01]$ and $\phi \in [\pi, \frac{3\pi}{2}]$. The components $G_1, G_2$ and $G_4$ are identically zero in the first quadrant of $B$, and

$$
\|A - B\| = \|A - C\| = \sqrt{4 + 2\sqrt{2}} > 1.01 + 1.6
$$

$$
= 1.01 + \max \|B - \partial \text{ supp } G_1\|
$$

$$
= 1.01 + \max \|C - \partial \text{ supp } G_2\|,
$$

so $F_\pi$ and $F_{3\pi}$ are identically zero for $x$ as above. Since $G_0(x) = F_\pi(x) + G_3(x)$, and we have:
Figure 11. $G_4$.

(I-1) $\frac{d}{d\phi} G_0(\mathbf{A} + (1, \phi)) > 0$, for $\phi \in [0, 1) \cup (1, 1.01]$, and

(I-2) $\frac{d}{d\phi} G_0(\mathbf{A} + (\rho, \phi)) < 0$ for $\rho \in (0.4, 1)$.

Near caldera $\mathbf{B}$, we have

$$G_0(x) = -F_\nu(x) - F_\nu(x) + G_1(x) + G_2(x) + G_4(x)$$

whenever $x = \mathbf{B} + (\rho, \phi)$ with $\rho \in [0, 1.01]$ and $\phi \in [-\frac{\pi}{7}, \frac{\pi}{7}]$. Again, this is true by a similar argument to (3.4). On the rim of caldera $\mathbf{B}$, $G_0$ is strictly increasing in the clockwise direction, i.e. $\frac{d}{d\phi} G_0(\mathbf{B} + (1, \phi)) > 0$. To see this, note that

(II-1) $\frac{d}{d\phi} F_\nu(\mathbf{B} + (1, \phi)) = 0$;

(II-2) $\frac{d}{d\phi} F_\nu(\mathbf{B} + (1, \phi)) \leq 0$ and is strictly positive whenever $\phi \in (-\frac{\pi}{7}, 0]$;

(II-3) $\frac{d}{d\phi} G_1(\mathbf{B} + (1, \phi)) \geq 0$ and is strictly positive whenever $\phi \in (0, \frac{\pi}{7}]$;

(II-4) $\frac{d}{d\phi} G_2(\mathbf{B} + (1, \phi)) \geq 0$;

(II-5) $\frac{d}{d\phi} G_4(\mathbf{B} + (1, \phi)) \geq 0$ and is strictly positive for $\phi = -\frac{\pi}{7}$.

For points less than one unit away from $\mathbf{B}$, that is, for points $\mathbf{B} + (\rho, \phi) \in B(\mathbf{B} + O)$, the radial slope is strictly negative:

$$\frac{d}{d\rho} G_0(\mathbf{B} + (\rho, \phi)) < 0$$

To see this, note that for such points:

(III-1) $\frac{d}{d\phi} F_\nu((\sqrt{2}, \pi) + (\rho, \phi)) > 0$;

(III-2) $\frac{d}{d\phi} F_\nu((\sqrt{2}, \pi) + (\rho, \phi)) \geq 0$;

(III-3) $\frac{d}{d\phi} G_1((\sqrt{2}, \pi) + (\rho, \phi)) \leq 0$;

(III-4) $\frac{d}{d\phi} G_2((\sqrt{2}, \pi) + (\rho, \phi)) \leq 0$;

(III-5) $\frac{d}{d\phi} G_4((\sqrt{2}, \pi) + (\rho, \phi)) \leq 0$.

Near caldera $\mathbf{C}$, we have a symmetric situation. At points of distance one from $\mathbf{C}$, $G_0$ is strictly increasing in the counterclockwise direction:

$$\frac{d}{d\phi} G_0(\mathbf{C} + (1, \phi)) < 0,$$
and for points less than one unit away from \( C \) the radial slope is strictly negative:

\[
\frac{d}{d\rho} G_0(C + (\rho, \phi)) < 0.
\]

The final piece to check is the set of points whose distance from \( \{A, B, C\} \) is greater than one. Let \( x \in R = B \setminus \{A, B, C\} + B \). Then the slope at \( x \) in the direction \( (1,1) \) is strictly positive:

\[
\nabla G_0(x) \cdot (1,1) > 0.
\]

We verify this fact as follows:

(IV-1) \( \nabla F_1(x) \cdot (1,1) \geq 0 \) and is strictly positive on \( R \cap (A + 1.5B) \);

(IV-2) \( \nabla G_3(x) \cdot (1,1) \geq 0 \) on \( R \cap (A + 1.5B) \);

(IV-3) \( \nabla (-F_2)(x) \cdot (1,1) \geq 0 \) and is strictly positive on \( R \cap (B + 1.5B) \);

(IV-4) \( \nabla G_1(x) \cdot (1,1) \geq 0 \), on \( R \cap (B + 1.5B) \);

(IV-5) \( \nabla (-F_{3/2} + G_2)(x) \cdot (1,1) \geq 0 \) and is strictly positive on \( R \cap (C + 1.5B) \);

(IV-6) \( \nabla G_2(x) \cdot (1,1) \geq 0 \), on \( R \cap (C + 1.5B) \); Since \( R \subset A, B, C + 1.5B \), we have accounted for all of \( R \).

3.4. Making \( G_0 \) Even on the Sphere. The constructed function \( G_0 \) has no critical points inside \( B \), but it requires some additional work in order to be made even on \( S \). We introduce a lemma that allows us to even out the boundary:

**Lemma 3.1.** Let \( \chi \in \mathbb{R} \), \( x = (\sqrt{2}, \chi) \), and \( L := B \cap (x + B) \) (see Figure 12). Let \( h \in C^1(\mathbb{R}) \) satisfy

1. \( \text{supp } h \subset [\chi - \frac{\pi}{4}, \chi + \frac{\pi}{4}] \);
2. \( h'(\chi - \frac{\pi}{4}) = h'(\chi + \frac{\pi}{4}) = 0 \);
3. \( h'(\phi) \geq 0 \) for \( \phi \in (\chi - \frac{\pi}{4}, \chi) \) and
4. \( h'(\phi) \leq 0 \) for \( \phi \in (\chi, \chi + \frac{\pi}{4}) \).

Then there exists a function \( H \in C^1(\mathbb{R}^2) \) such that:

1. \( H(s(\phi)) = h(\phi) \) for all \( \phi \in [\chi - \frac{\pi}{4}, \chi + \frac{\pi}{4}] \);
2. \( \frac{d}{d\rho} H(x + (\rho, \psi)) \leq 0 \) for \( \rho \leq 1 \) and \( \psi \in [-\chi - \frac{\pi}{2}, -\chi + \frac{\pi}{2}] \) (that is, in the right sector of \( x + B \) containing \( L \));
3. \( H(x) = 0 \) for all \( x \notin (1, \infty) L \).

**Proof.** Let \( f \in C^1[0, +\infty) \) satisfy \( f(0) = f'(0) = f'(1) = 0 \), \( f[1, +\infty) = 1 \) and \( f' \geq 0 \). For any \( \phi \in [\chi - \frac{\pi}{4}, \chi + \frac{\pi}{4}] \), define

\[
p(\phi) := \sqrt{\frac{2}{\cos(\phi - \chi) - \sqrt{\cos^2(\phi - \chi)}}}.
\]

This defines \( p(\phi) \) to equal \( \min\{\rho > 0 : (\rho, \phi) \in L\} \); note that \( p(\phi) \in [\sqrt{2} - 1, 1] \) whenever \( \phi \in (\chi - \frac{\pi}{4}, \chi + \frac{\pi}{4}) \) (see Figure 13). Now let:

\[
H((\rho, \phi)) := \begin{cases} 
\frac{h(\phi) f(\frac{p(\phi)}{p(\phi) + 1})}{p(\phi)} & (\rho, \phi) \in (1, \infty) L \\
0 & \text{otherwise}
\end{cases}
\]

Note that \( H((p(\phi), \phi)) = 0 \) and \( H((1, \phi)) = h(\phi) \) for all \( \phi \in [\chi - \frac{\pi}{4}, \chi + \frac{\pi}{4}] \). The definition guarantees that \( H \) is \( C^1 \) everywhere except possibly on the boundary of the two sets; on the other side, it is directly checked that \( H' \) equals zero at all points of the boundary and is continuous there (due to the facts that \( f(0) = f'(0) = 0 \) and \( h(\chi) = h(\chi + \frac{\pi}{2}) = h'(\chi) = h'(\chi + \frac{\pi}{2}) = 0 \)). Then (1) and (3) follow from the definition; to see (2), note that, for a fixed \( \psi \), \( H(x + (\rho, \psi)) \) is non-increasing.
in direction $\rho$ since $f' \geq 0$, $h'(\phi) \geq 0$ for $\phi \in (\chi - \frac{\pi}{4}, \chi)$ and $h'(\phi) \leq 0$ for $\phi \in (\chi, \chi + \frac{\pi}{4})$.

As an example of how Lemma 3.1 works, Figure 14 shows the graph of $H$ when $h(\phi) = \cos^2(2\phi)$ and $\chi = 0$. We will use $H$ to smoothly reshape the function $G_0$ on one portion of the sphere at a time, leaving values outside the associated lunette unchanged.

Define $g_0$ by $g_0(\phi) := G_0((1, \phi))$. Then $g_0 \in C^1$ is a $2\pi$-periodic function that traces $G_0$ over the sphere $S$ (Figure 15). It is readily checked that

$$g_0(\phi) > 0 \text{ for } \phi \in \left[0, \frac{\pi}{2}\right],$$

$$g_0(\phi) < 0 \text{ for } \phi \in \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right],$$
Figure 14. graph of $H$, with $h(\phi) = \cos^2(2\phi)$ and $\chi = 0$.

Figure 15. $g_0$.

$g_0'(\phi) > 0$ for $\phi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right) \cup \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$, and

$g_0'(\phi) < 0$ for $\phi \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$.

The reflective definition of $G_2$ and the overall symmetry guarantees that

$g_0\left(\frac{5\pi}{4}, -\phi\right) = g_0\left(\frac{5\pi}{4} + \phi\right)$ for $\phi \in \left[0, \frac{\pi}{2}\right]$.

(3.5)

$-\frac{d}{d\phi} g_0\left(\frac{3\pi}{4}, -\phi\right) \geq \frac{d}{d\phi} g_0\left(\frac{3\pi}{4}, +\phi\right)$ for $\phi \in \left[0, \frac{\pi}{4}\right]$.

(3.6)

Similarly,

$\frac{d}{d\phi} g_0\left(\frac{7\pi}{4}, +\phi\right) \geq -\frac{d}{d\phi} g_0\left(\frac{7\pi}{4}, -\phi\right)$ for $\phi \in \left[0, \frac{\pi}{4}\right]$.

(3.7)

Denote
NO CRITICAL POINTS

1. $M := g_0(0) > 0$,
2. $m := g_0\left(\frac{\pi}{2}\right) > 0$,
3. $l := g_0\left(\frac{3\pi}{4}\right) = g\left(\frac{5\pi}{4}\right) < 0$ and
4. $L := g_0\left(\frac{5\pi}{4}\right) < 0$.

Then $M$ is the greatest height of the ridge on caldera $A$, $m$ is the lowest height on that ridge, $l$ is the depth of the shallowest point of the gorges on the inverted calderas $B$ and $C$, and $L$ is the depth of the deepest point (Figure 15). We want to use Lemma 3.1 to remodel $G_0$ along its rim, making the function $\pi$-periodic along the rim (i.e. even on the sphere), without adding any critical points to the function.

Choose some $\pi$-periodic $g \in C^1(\mathbb{R})$, such that:

\[ g(0) = M, \quad g\left(\frac{\pi}{4}\right) = L \text{ and } g(\phi) = g_0(\phi) \text{ for } \phi \in \left[-\frac{\pi}{4}, 0\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right], \]

\[ g(\phi) = g_0(\phi) \geq g_0(\phi + \pi) \text{ for } \phi \in \left[-\frac{\pi}{4}, 0\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right], \]

\[ g'(\phi) \leq g'_0(\phi) \text{ for } \phi \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, 7\pi\right), \]

\[ g'(\phi) \geq g'_0(\phi) \text{ for } \phi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right) \cup \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right) \]

(we use here (3.5), (3.6), (3.7)), and

\[ g_0(\phi + \pi) \leq g(\phi) \leq g_0(\phi) \text{ for } \phi \in \left[0, \frac{\pi}{2}\right]. \]

This is possible since here $g_0(\phi + \pi) < 0 < g_0(\phi)$ and

\[ g_0'(\frac{j\pi}{4}) = 0 \text{ for all } j \in \mathbb{N}. \]

The last implies also that

\[ g(\phi) \geq g_0(\phi) \text{ for } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]. \]

We now have a $\pi$-periodic function $g$ that agrees with $g_0$ on $\left[-\frac{\pi}{4}, 0\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$, and we wish to make $g_0$ agree with $g$ on the remainder of the unit circle. To this end,
we use (3.8), (3.9) and apply Lemma 3.1 three times to get $H_1, H_2, H_3 \in C^1(\mathbb{R}^2)$, using

$$\chi_1 := \frac{\pi}{4}, \chi_2 := \pi \text{ and } \chi_3 := \frac{3\pi}{2};$$
$$x_1 := A, \ x_2 := B \text{ and } x_3 := C;$$

and

$$h_i := (g - g_0)I_{\chi_i - \frac{\pi}{4}, \chi_i + \frac{\pi}{4}}; \quad i = 1, 2, 3.$$ 

In each of the three cases, $h_i \in C^1(\mathbb{R})$ since (3.10) and the definition of $g$ assure us that $h_i'(\chi_i - \frac{\pi}{4}) = h_i'(\chi_i + \frac{\pi}{4}) = h_i'(\chi_i + \frac{\pi}{2}) = 0$. As a result, we obtain functions $H_i \in C^1(\mathbb{R}^2)$, $i = 1, 2, 3$, such that:

(V-1) $H_1(s(\phi)) = (g - g_0)(\phi)$ for all $\phi \in [0, \frac{\pi}{4}]$;
(V-2) $H_2(s(\phi)) = (g - g_0)(\phi)$ for all $\phi \in [\frac{\pi}{4}, \frac{2\pi}{3}]$;
(V-3) $H_3(s(\phi)) = (g - g_0)(\phi)$ for all $\phi \in [\frac{2\pi}{3}, \frac{3\pi}{4}]$;
(V-4) $\frac{\partial}{\partial \rho} H_i(x + (\rho, \psi)) \geq 0$ for $\rho \leq 1$ and $\psi \in [\pi, \frac{3\pi}{4}]$;
(V-5) $\frac{\partial}{\partial \rho} H_i(x + (\rho, \psi)) \leq 0$ for $\rho \leq 1$ and $\psi \in [-\pi - \frac{\pi}{4}, -\chi_i + \frac{\pi}{4}], \quad i = 2, 3$;
(V-6) $H_i(x) = 0$ for all $x \notin (1, \infty)(B \cap (x_i + B)), \quad i = 1, 2, 3$.

Let $G := G_0 + H_1 + H_2 + H_3$. Then the previous calculations for $G_0$ and the properties of $H_i$ above guarantee that $G$ has no critical points.

Let $\tilde{g}(\phi) = G(1, \phi)$. Then

$$\tilde{g}(\phi) = G_0(\phi) + H_1(\phi) + H_2(\phi) + H_3(\phi)$$
$$= g_0(\phi) + (g - g_0)I_{[0, \frac{\pi}{4}]} + I_{[\frac{\pi}{4}, \frac{2\pi}{3}]} + I_{[\frac{2\pi}{3}, \frac{3\pi}{4}]}(\phi)$$
$$= g(\phi),$$

so $G$ is even on the sphere.

4. **Even-valuedness on a Neighbourhood of $S$**

In the above example, it is not possible to achieve even-valuedness on an open neighborhood of the sphere. In fact, if $G$ is a function that is even on a neighborhood of $S$, then it has to have a critical point in the ball, as we will see in Proposition 4.1, using some elementary degree theory [GK]. If $v : U \rightarrow \mathbb{R}^2 \backslash \{0\}$ is a continuous vector field (defined in an open set $U \subset \mathbb{R}^2$) and $C$ is a smooth oriented curve in $U$, then $\text{rot}(v, C)$ equals the number of counter-clockwise revolutions of $v(x)$ whenever $x$ passes through $C$ (in its positive direction). When $C$ is a closed curve, then of course $\text{rot}(v, C) \in \mathbb{N}$. Further, if $C$ is contractible and $v$ does not vanish in the interior of $C$, then $\text{rot}(v, C) = 0$. We may now prove:

**Proposition 4.1.** Let $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a $C^1$-function defined on a neighborhood $U$ of $B$. If $f$ is even in some neighborhood $W$ of $S$, then there is some $x_0 \in B$ such that $\nabla f(x_0) = 0$.

**Proof.** Suppose $v = \nabla f$ does not vanish on $B_1$. Let $C = S$, oriented anticlockwise, and $C'$ be the part of $C$ lying in the closed upper semiplane. Since $f$ is even on $W$, $v$ is odd there; now $v(1, 0) = v(-1, 0)$ implies $\text{rot}(v, C') = n + \frac{1}{2}, \; n \in \mathbb{N}$, while the fact that for any $x \in C'$, $v(x) = -v(-x)$ implies that $\text{rot}(v, C') = \text{rot}(v, -C')$ and hence $\text{rot}(v, C) = \text{rot}(v, C') + \text{rot}(v, -C') = 2n + 1$. But since $v$ does not vanish on $B_1$ and $C$ is contractible, we have $\text{rot}(v, C) = 0$, a contradiction. \qed
NO CRITICAL POINTS

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