On the Fractional Integrals of the Weierstrass Functions: the Exact Box Dimension\footnote{Supported in part by National and Zhejiang Provincial Natural Science Foundations of China, and by Ningbo Key Doctoral Funds. The main part of the work contained in this paper was done while the first named author was visiting Simon Fraser University, Canada, he thanks Dr. P. B. Borwein and the Department of Mathematics of Simon Fraser University for their hospitality.}

S. P. Zhou  
The Institute of Mathematics, Ningbo University,  
Ningbo, Zhejiang 315211 China

K. Yao  
Department of Mathematics, Zhejiang University,  
Hangzhou, Zhejiang 310028 China

**Abstract.** The present paper investigates the fractal structure of the fractional integrals of the Weierstrass functions. The exact box dimension for many important cases is established. We need to point out that, although the result itself achieved in the present paper is interesting, the new technique and method should be emphasized. These novel ideas could be useful to establish the box dimension or Hausdorff dimension (especially for the lower bounds) for more general groups of functions.

**Key Words.** fractal, fractional calculus, Weierstrass function, box dimension

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§1. Introduction

As a new branch in mathematics, fractal geometry has shown its value and influence with wide important applications over many aspects. Some initial and conclusive works on fractals and its mathematical foundations were done in [3], [5], [11]. Especially, because of their special fractal structures, the Weierstrass type functions

\[ W(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^k t \]

for $1 < s < 2$ and $\lambda > 1$, and, in general,

\[ B(t) = \sum_{k=1}^{\infty} \lambda_k^{s-2} \sin \lambda_k t \]
for $1 < s < 2$ and $\lim_{k \to \infty} \lambda_k = \infty$ have been investigated extensively, and details on their graphs and fractal dimensions have been given in [1]-[4], [9], [11].

On the other hand, fractional calculus attracts more and more attentions recently. Some important and detailed works were achieved in [7], [8]. We cite a fundamental conception as follows.

**Definition ([7]).** Let $f$ be a function piecewisely continuous on $(0, \infty)$ and integrable on any finite subinterval of $(0, \infty)$. Then

$$D^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t - \xi)^{v-1} f(\xi) d\xi$$

is called the fractional integral of $f$ of order $v$ for $t > 0$ and $\text{Re}(v) > 0$.

It is generally acknowledged that there exists some relationship between fractal and fractional calculus. Fractional calculus was applied by Mandelbrot and Vanness [6] to deal with the fractal noise (white noise), while was applied by Tatom [10] to the Koch curve.

The present paper investigates the fractal structure of the fractional integrals of the Weierstrass functions. The exact box dimension for many important cases is established. We arrange the paper by the following order: A brief introduction of the fractional integral functions of the Weierstrass functions $W(t)$ is given in Section 2, while their graphs, with graphs of the Weierstrass functions, are presented in Section 3. Finally, the main results will be given in Section 4 and the proofs be processed in Section 5.

We need to point out that, although the result itself achieved in the present paper is interesting, the new technique and method should be emphasized. These novel ideas could be useful to establish the box dimension or Hausdorff dimension (especially for the lower bounds) for more general groups of functions.

The box dimension of the fractional derivatives of $W(t)$ could be calculated similarly.

**§2. The Fractional Integrals**

We begin with the fractional integrals of two basic trigonometric functions $\sin \lambda t$ and $\cos \lambda t$. By the definition, for $t > 0$ and $\text{Re}(v) > 0$,

$$D^{-v} \sin \lambda t = \frac{1}{\Gamma(v)} \int_0^t (t - \xi)^{v-1} \sin \lambda \xi d\xi =: S_t(v, \lambda),$$

$$D^{-v} \cos \lambda t = \frac{1}{\Gamma(v)} \int_0^t (t - \xi)^{v-1} \cos \lambda \xi d\xi =: C_t(v, \lambda).$$

The relationship between $S_t(v, \lambda)$ and $C_t(v, \lambda)$ is summarized into the following
Lemma 1 ([7], [8]). For \( S_t(v, \lambda) \) and \( C_t(v, \lambda) \), we have the following relations:

(a) \( S_t(v, \lambda) = \lambda C_t(v + 1, \lambda) \);
(b) \( d(S_t(v, \lambda))/dt = S_t(v - 1, \lambda) \);
(c) \( d(C_t(v, \lambda))/dt = C_t(v - 1, \lambda) \).

Finally, for \( 1 < s < 2, \lambda > 1 \) and \( 0 < v < 1 \), define

\[
g(t) := D^{-v}(W(t)) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} S_t(v, \lambda^k)
\]

(1)

to be the fractional integrals of \( W(t) \).

§3. Graphs

Four graphs are presented in this section. The graphs of \( W(t) \) in case \( s = 1.5, \lambda = 1.5 \) and in case \( s = 1.7, \lambda = 2 \) are numbered as Fig 1 and 2, while the graphs of \( g(t) \) in case \( s = 1.5, \lambda = 1.5, v = 0.2 \) and in case \( s = 1.7, \lambda = 2, v = 0.6 \) are numbered as Fig 3 and 4.

§4. Main Results

Set \( I = [0,1] \). In the present paper, we always use Graph(\( f, I \)) to stand for the graph of \( f(t) \) over \( I \), and \( \dim_B F \) for the box dimension of a subset \( F \) of \( \mathcal{R}^2 \). In addition, \( \overline{\dim}_B F \) and \( \underline{\dim}_B F \) indicate the upper and lower box dimension of \( F \) respectively. Interested readers could find the details in [3].

Theorem 1. Let \( 1 < s < 2, \lambda > 1, 0 < v < 1 \), and \( g(t) \) be the fractional integral function of \( W(t) \) of order \( v \) as defined in (1), then

\[
\overline{\dim}_B \text{Graph}(g, I) \leq \begin{cases} 
  s - v, & s > 1 + v, \\
  1, & s \leq 1 + v.
\end{cases}
\]

Theorem 2. Let \( 1 < s < 2, 0 < v < 1, s > 1 + v, g(t) \) be the fractional integral function of \( W(t) \) of order \( v \) as defined in (1), then for sufficiently large \( \lambda > 1 \), it holds that

\[
\dim_B \text{Graph}(g, I) \geq s - v.
\]

A combination of Theorem 1 and Theorem 2 leads to very important particular cases which we write as the following

Theorem 3. Let \( 1 < s < 2, 0 < v < 1, s > 1 + v, g(t) \) be the fractional integral function of \( W(t) \) of order \( v \) as defined in (1), then for sufficiently large \( \lambda > 1 \), it holds that

\[
\dim_B \text{Graph}(g, I) = s - v.
\]
For simplicity, any \( C \) or \( O(1) \) in the present paper indicates a positive constant that may depend on \( v \) and may have different values at different occurrences (even at the same line). We also use other symbols to indicate constants if needed.

§5. Proofs

The following Lemma 2 is a clear fact.

**Lemma 2.** Let \( 0 < \alpha < 1 \), then \( f(x) = x^{-\alpha} \) is decreasing and convex on \((0, \infty)\).

Write

\[
\tilde{S}_t(v, \lambda) = \Gamma(v)S_t(v, \lambda), \quad \tilde{C}_t(v, \lambda) = \Gamma(v)C_t(v, \lambda).
\]

**Lemma 3.** Let \( 0 < v < 1 \), \( \lambda > 1 \) be a positive number, \( 0 < h < 1 \), \( h_0 = \max\{h, \lambda^{-1}\} \), then for \( t > 4h_0 \) we have

\[
\left| \tilde{S}_{t+h}(v, \lambda) - \tilde{S}_t(v, \lambda) \right| \leq C_1 h_0^v.
\]

**Proof.** Evidently, it holds that

\[
\int_t^{t+h} (t + h - \xi)^{v-1} \sin \lambda \xi d\xi = O(h^v).
\]

At the same time, it is also clear that

\[
\int_{t-2h_0}^{t} (t + h - \xi)^{v-1} \sin \lambda \xi d\xi - \int_{t-2h_0}^{t} (t - \xi)^{v-1} \sin \lambda \xi d\xi = O(h_0^v).
\]

By setting \( \frac{2N_0}{\lambda} \pi \leq t - 2h_0 < \frac{2(N_0+1)}{\lambda} \pi \) for some \( N_0 \), we calculate that

\[
J(t, h) := \int_0^{t-2h_0} ((t + h - \xi)^{v-1} - (t - \xi)^{v-1}) \sin \lambda \xi d\xi
\]

\[
= \sum_{j=0}^{N_0-1} \int_{2j \pi / \lambda}^{(2j+1) \pi / \lambda} ((t + h - \xi)^{v-1} - (t - \xi)^{v-1}) \sin \lambda \xi d\xi + O(h_0^v)
\]

\[
= \sum_{j=0}^{N_0-1} \int_{2j \pi / \lambda}^{(2j+1) \pi / \lambda} ((t + h - \xi)^{v-1} - (t - \xi)^{v-1} - (t + h - \xi - \pi / \lambda)^{v-1} + (t - \xi - \pi / \lambda)^{v-1}) \sin \lambda \xi d\xi + O(h_0^v)
\]

(setting \( \xi = u + \pi / \lambda \))

\[
= \sum_{j=0}^{N_0-1} \int_{2j \pi / \lambda}^{(2j+1) \pi / \lambda} (t - \xi)^{v-1} \left( \left(1 + \frac{h}{t - \xi}\right)^{v-1} - 1 - \left(1 + \frac{h - \pi / \lambda}{t - \xi}\right)^{v-1} \right)
\]
\[
+ \left(1 - \frac{\pi/\lambda}{t - \xi}\right)^{v-1} \sin \lambda \xi d\xi + O(h_0^v).
\]

Since\(^2\) for small \(0 < \beta_1, \beta_2 < 1\),
\[
(1 + \beta_1)^{v-1} - 1 - (1 + \beta_1 - \beta_2)^{v-1} + (1 - \beta_2)^{v-1} \approx \beta_1 \beta_2,
\]
it follows that
\[
|J(t, h)| \leq C h \lambda^{-1} \sum_{j=0}^{N_0-1} \left| \int_{2j\pi/\lambda}^{(2j+1)\pi/\lambda} (t - \xi)^{v-3} \sin \lambda \xi d\xi \right| + O(h_0^v)
\]
\[
\leq C h \lambda^{-1} \sum_{j=0}^{N_0-1} \int_{2j\pi/\lambda}^{(2j+1)\pi/\lambda} (t - \xi)^{v-3} d\xi + O(h_0^v)
\]
\[
\leq C h \lambda^{-1} \int_{0}^{t-2h_0} (t - \xi)^{v-3} d\xi + O(h_0^v)
\]
\[
\leq C h h_0^{v-2} \lambda^{-1} + O(h_0^v),
\]
with (2), (3), Lemma 3 is proved.

We still need the following more careful treatment for \(\tilde{S}_{t+h}(v, \lambda) - \tilde{S}_t(v, \lambda)\) in some special cases.

**Lemma 4.** Let \(0 < v < 1, \lambda > 1\) be a sufficiently large number, \(h_\lambda = 4\pi/\lambda\), \(t_j = jh_\lambda, j = 5, 6, \ldots, h_1 = 3\pi/\lambda\), then
\[
\tilde{S}_{t_j+h_1}(v, \lambda) - \tilde{S}_{t_j}(v, \lambda) \geq C_2 \lambda^{-v}.
\]

**Proof.** We check that,
\[
\tilde{S}_{t_j+h_1}(v, \lambda) - \tilde{S}_{t_j}(v, \lambda) = \int_{t_j}^{t_j+h_1} (t_j + h_1 - \xi)^{v-1} \sin \lambda \xi d\xi
\]
\[
+ \int_{t_j-2h_1}^{t_j} \left( (t_j + h_1 - \xi)^{v-1} - (t_j - \xi)^{v-1} \right) \sin \lambda \xi d\xi
\]
\[
+ \int_{0}^{t_j-2h_1} \left( (t_j + h_1 - \xi)^{v-1} - (t_j - \xi)^{v-1} \right) \sin \lambda \xi d\xi =: I_1 + I_2 + I_3.
\]}

Now by Lemma 2,
\[
I_1 = \int_{4j\pi/\lambda}^{(4j+3)\pi/\lambda} (t_j + h_1 - \xi)^{v-1} \sin \lambda \xi d\xi
\]
\[
= \int_{4j\pi/\lambda}^{(4j+1)\pi/\lambda} (t_j + h_1 - \xi)^{v-1} \sin \lambda \xi d\xi
\]
\[\text{As usual, } A \approx B \text{ means that there is a constant } C > 0 \text{ such that } C^{-1} \leq |A/B| \leq C.\]
+ \int_{(4j+2)\pi/\lambda}^{(4j+1)\pi/\lambda} \left( (t_j + h_1 - \xi)^{v-1} - (t_j + h_1 - \xi - \pi/\lambda)^{v-1} \right) \sin \lambda \xi d\xi \geq 0,$

while by Lemma 2 again,

$$I_2 = \int_{(4j-6)\pi/\lambda}^{4j\pi/\lambda} \left( (t_j + h_1 - \xi)^{v-1} - (t_j - \xi)^{v-1} \right) \sin \lambda \xi d\xi$$

$$= \sum_{k=2j-3}^{2j-1} \int_{2k\pi/\lambda}^{(2k+1)\pi/\lambda} \left( (t_j + h_1 - \xi)^{v-1} - (t_j - \xi)^{v-1} - (t_j + h_1 - \xi - \pi/\lambda)^{v-1} + (t_j - \xi - \pi/\lambda)^{v-1} \right) \sin \lambda \xi d\xi \geq 0.$$

Finally, similar to the estimation of $J(t, h)$ in Lemma 1, with (4), for sufficiently large $\lambda$, we get

$$I_3 \geq C h_1 \lambda^{-1} \sum_{k=0}^{2j-4} \int_{2k\pi/\lambda}^{(2k+1)\pi/\lambda} (t_j - \xi)^{v-3} \sin \lambda \xi d\xi$$

$$\geq C h_1 \lambda^{-1} \sum_{k=0}^{2j-4} \int_{2k\pi/\lambda}^{(2k+3/4)\pi/\lambda} (t_j - \xi)^{v-3} \sin \lambda \xi d\xi$$

$$\geq C h_1 \lambda^{-1} \sum_{k=0}^{2j-4} \int_{2k\pi/\lambda}^{(2k+3/4)\pi/\lambda} (t_j - \xi)^{v-3} d\xi$$

$$\geq C h_1 \lambda^{-1} \sum_{k=0}^{2j-4} \left( \left( \frac{2k + 3/4}{\lambda} \right)^{v-2} - \left( \frac{2k + 1/4}{\lambda} \right)^{v-2} \right)$$

$$\geq C h_1 \lambda^{-1} \left( \left( \frac{8 - 3/4}{\lambda} \right)^{v-2} - \left( \frac{8 - 1/4}{\lambda} \right)^{v-2} \right) \geq C_2 \lambda^{-v}.$$

Combining all the above estimates, we have the required result.

**Lemma 5.** Let $0 < v < 1$, $\lambda > 1$ be a positive number, $0 < h < 1$, $h_0 = \max \{h, \lambda^{-1}\}$, then for any fixed $\eta \in (0, h)$ and $t > 4h_0$, we have

$$\left| \tilde{S}_{t+\eta}^\prime(v, \lambda) \right| \leq C_3 \lambda h_0^v.$$

**Proof.** As we know (by Lemma 1), $\tilde{S}_t^\prime(v, \lambda) = \lambda \tilde{C}_t^\prime(v + 1, \lambda) = \lambda \tilde{C}_t(v, \lambda)$. Hence

$$\tilde{S}_{t+\eta}^\prime(v, \lambda) = \lambda \int_0^{t+\eta} (t + \eta - \xi)^{v-1} \cos \lambda \xi d\xi.$$

In a way similar to the proof of Lemma 3, by choosing $N_1$ to satisfy $\frac{2N_1 + 1/2}{\lambda} \leq t + \eta - 2h_0 < \frac{2(N_1 + 1) + 1/2}{\lambda}$ and writing

$$\int_0^{t+\eta} (t + \eta - \xi)^{v-1} \cos \lambda \xi d\xi = \int_{t+\eta - 2h_0}^{t+\eta} (t + \eta - \xi)^{v-1} \cos \lambda \xi d\xi$$

$$= \sum_{k=0}^{N_1} \int_{2k\pi/\lambda}^{(2k+1)\pi/\lambda} \left( (t + \eta - \xi)^{v-1} - (t + \eta - \xi - \pi/\lambda)^{v-1} \right) \sin \lambda \xi d\xi \geq 0,$$
\[
\begin{align*}
&\quad + \int_{(2N+1/2)\pi/\lambda}^{t+\eta-2\lambda} (t + \eta - \xi)^{v-1} \cos \lambda \xi \, d\xi + \int_0^{\pi/(2\lambda)} (t + \eta - \xi)^{v-1} \cos \lambda \xi \, d\xi \\
&\quad + \sum_{j=0}^{N-1} \int_{(2j+1/2)\pi/\lambda}^{(2j+3/2)\pi/\lambda} \left( (t + \eta - \xi)^{v-1} - (t + \eta - \xi - \pi/\lambda)^{v-1} \right) \cos \lambda \xi \, d\xi,
\end{align*}
\]

we see that

\[
|\tilde{S}_{t,\eta}^v(v, \lambda)| \leq C \lambda \sum_{j=0}^{N-1} \int_{(2j+1/2)\pi/\lambda}^{(2j+3/2)\pi/\lambda} (t + \eta - \xi)^{v-1} \left| 1 - \left( 1 - \frac{\pi/\lambda}{t + \eta - \xi} \right)^{v-1} \right| \, d\xi + O(\lambda h_0^v)
\]

\[
\leq C \int_0^{t+\eta-2\lambda} (t + \eta - \xi)^{v-2} \, d\xi + O(\lambda h_0^v) \leq Ch_0^{v-1} + O(\lambda h_0^v) \leq C_3 \lambda h_0^v
\]

by a similar argument to that of Lemma 5. Thus we have the required result.

Let \( C_{[0,1]} \) be the space of all continuous functions on the interval \([0, 1]\). As usual, \( \omega(f, t) \) is the modulus of continuity of \( f \) on \([0, 1]\).

**Lemma 6.** Let \( 1 < s < 2, \lambda > 1 \) and \( 0 < v < 1 \), then

\[
\omega(g, h) \leq \begin{cases} 
Ch^{-s+v+2}, & s > 1 + v, \\
Ch|\log h|, & s = 1 + v, \\
Ch, & s < 1 + v.
\end{cases}
\]

**Proof.** For \( 0 < h < 1 \), choose \( N \) such that\( ^34\pi \lambda^{-(N+1)} < h \leq 4\pi \lambda^{-N} \). Write

\[
g_n(t) = \sum_{k=1}^n \lambda^{(s-2)k} S_t(v, \lambda^k).
\]

Then by the mean value theorem, there is an \( \eta \in (0, h) \) such that

\[
g_{N-1}(t + h) - g_{N-1}(t) = g'_{N-1}(t + \eta) h = h \sum_{k=1}^{N-1} \lambda^{(s-2)k} S'_{t+\eta}(v, \lambda^k).
\]

Then it immediately follows from Lemma 5 that (note that \( h_0 \approx \lambda^{-k} \) in this case)

\[
|g_{N-1}(t + h) - g_{N-1}(t)| \leq Ch \sum_{k=1}^{N-1} \lambda^{(s-v-1)k}
\]

\[
\leq \begin{cases} 
Ch^{(s-v-1)(N-1)} \leq Ch^{-s+v+2}, & s > 1 + v, \\
ChN \leq Ch|\log h|, & s = 1 + v, \\
Ch, & s < 1 + v.
\end{cases}
\]

\(^3\)We set \( 4\pi \lambda^{-(N+1)} < h \leq 4\pi \lambda^{-N} \) for the sake of convenience to fit in with the conditions of the following Lemma 7.
By Lemma 3, it is clear that (note that \( h_0 \approx \lambda^{-N} \approx h \) in this case)

\[
|S_{t+h}(v, \lambda^N) - S_t(v, \lambda^N)| \leq C\lambda^{-vN}.
\]  

(6)

At the same time (note that \( h_0 \approx \lambda^{-N} \approx h \) in this case), from Lemma 3 we get

\[
\left| \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} (S_{t+h}(v, \lambda^k) - S_t(v, \lambda^k)) \right| \leq C h^v \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} \leq C h^{-s+v+2}.
\]

(7)

Combining (5)-(7), with

\[
g(t) = g_{N-1}(t) + \lambda^{(s-2)N} S_t(v, \lambda^N) + \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} S_t(v, \lambda^k),
\]

we have the required inequalities.

**Remark.** We should note that, it can be actually deduced from (5) that

\[
|g_{N-1}(t+h) - g_{N-1}(t)| \leq C_4 \lambda^{v+1-s} \lambda^{(s-v-2)N} \quad (5')
\]

in case \( s > 1 + v \), and from (7) that

\[
\left| \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} (S_{t+h}(v, \lambda^k) - S_t(v, \lambda^k)) \right| \leq C h^v \frac{\lambda^{(s-2)(N+1)}}{1 - \lambda^{s-2}} \leq C_5 \lambda^{s-2} \lambda^{(s-v-2)N}. \quad (7')
\]

**Lemma 7.** Let \( 1 < s < 2, \ 0 < v < 1, \ \lambda > 1 \) be a sufficiently large number, \( h := h_{\lambda N} = 4\pi \lambda^{-N}, \ t_j = jh, \ j = 5, 6, \cdots, \ h_1 = 3\pi \lambda^{-N} \), then for \( s > 1 + v \) we have

\[
g(t_j + h_1) - g(t_j) \geq C h^{-s+v+2}.
\]

**Proof.** First note, as we indicated in the footnote of Lemma 6, this \( h \) satisfies the requirements in the proof of Lemma 6. By Lemma 4, (5') and (7'), we have

\[
g(t_j + h_1) - g(t_j) \geq \lambda^{(s-2)N} \left( S_{t_j+h_1}(v, \lambda^N) - S_{t_j}(v, \lambda^N) \right)
\]

\[
- \left| g_{N-1}(t_j + h_1) - g_{N-1}(t_j) \right| - \left| \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} (S_{t_j+h_1}(v, \lambda^k) - S_{t_j}(v, \lambda^k)) \right|
\]

\[
\geq \lambda^{(s-v-2)N} \left( C_2 - C_4 \lambda^{v+1-s} - C_5 \lambda^{s-2} \right) \geq C h^{-s+v+2}
\]

for sufficiently large \( \lambda \). Lemma 7 is proved.

Set

\[
R_f(t_1, t_2) = \sup_{t_1 \leq t, u \leq t_2} |f(t) - f(u)|.
\]
Lemma 8 ([3, Proposition 11.1]). Let \( f \in C_{[0,1]} \), \( 0 < \delta < 1 \), and \( m \) be the least integer greater than or equal to \( 1/\delta \). If \( N_\delta \) is the number of the squares of the \( \delta \)-mesh that intersects graph \( f \), then

\[
\delta^{-1} \sum_{j=0}^{m-1} R_f(j\delta, (j+1)\delta) \leq N_\delta \leq 2m + \delta^{-1} \sum_{j=0}^{m-1} R_f(j\delta, (j+1)\delta).
\]

Proof of Theorem 1. From Lemma 6 and Lemma 8,

\[
N_\delta \leq \begin{cases} 
C\delta^{-s+v}, & s > 1 + v, \\
C\delta^{-1}\log\delta, & s = 1 + v, \\
C\delta^{-1}, & s < 1 + v.
\end{cases}
\]

Theorem 1 then follows from a standard argument (see, for example, [3, Proposition 4.1]).

Proof of Theorem 2. Application of Lemma 7 and Lemma 8 yields that \( N_\delta \geq C\delta^{-s+v} \) for \( s > 1 + v \) and \( \delta = h = 4\pi \lambda^{-N} \) in case \( N \) is large enough. For arbitrary \( \delta, 4\pi \lambda^{-N-1} \leq \delta < 4\pi \lambda^{-N} \), a standard argument suggests the same conclusion still holds. Then it is clear that

\[
\dim_B \text{Graph}(g, I) \geq s - v
\]

holds in this case.

References


