Approximation by Rational Functions with Prescribed Numerator Degree in $L^p$ Spaces for $1 < p < \infty$

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ABSTRACT. The present paper establishes a complete result on approximation by rational functions with prescribed numerator degree in $L^p$ spaces for $1 < p < \infty$ and proves that if $f(x) \in L^p_{[-1,1]}$ changes sign exactly $l$ times in $(-1, 1)$, then there exist a $r(x) \in R_n^l$ such that

$$\|f(x) - r(x)\|_{L^p} \leq C_{p,l,b} \omega(f, n^{-1})_{L^p},$$

where $R_n^l$ indicates all rational functions whose denominators consist of polynomials of degree $n$ and numerators polynomials of degree $l$, and $C_{p,l,b}$ is a positive constant only depending on $p$, $l$ and $b$ which relates to the distance among the sign change points of $f(x)$ and will be given in §3.

Key Words. rational function, prescribed numerator, the Steklov function, the modified Jackson kernel

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§1. Introduction

Let $L^p_{[-1,1]}$, $1 \leq p < \infty$, be the space of $p$ power integrable functions on the interval $[-1, 1]$ with norm

$$\|f\|_{L^p_{[-1,1]}} = \left\{ \int_{-1}^{1} |f(x)|^p dx \right\}^{\frac{1}{p}},$$

and $\omega(f, \delta)_{L^p_{[-1,1]}}$ be the modulus of continuity in $L^p$ norm of $f \in L^p_{[-1,1]}$, that is,

$$\omega(f, \delta)_{L^p_{[-1,1]}} = \sup_{0 < h \leq \delta} \left\{ \int_{-1}^{1-} |f(x + h) - f(x)|^p dx \right\}^{\frac{1}{p}}.$$
For convenience, write
\[
\omega(f, \delta)_{L^p_{[-1,1]}} = \omega(f, \delta)_{L^p}, \quad \|f\|_{L^p_{[-1,1]}} = \|f\|_{L^p}.
\]
Denote by \(\Pi_n\) all polynomials of degree \(n\), and \(R^l_n\) all rational functions whose denominators consist of polynomials of degree \(n\) and numerators polynomials of degree \(l\), that is,
\[
R^l_n = \left\{ \frac{q(x)}{p(x)} : p(x) \in \Pi_n, q(x) \in \Pi_l \right\}.
\]

The subject of approximation by reciprocals of polynomials with real coefficients is a special field in rational approximation, and has attracted great interest. General speaking, the research on this topic is hard since reciprocals is not even closed for addition. To consider approximation by reciprocals, we must assume that the given function has a fixed sign on the given interval. There are some interesting results. The Jackson estimation in continuous function space was established by Leviatan, Levin and Saff [4], and was generalized to general \(L^p\) spaces for \(1 \leq p \leq \infty\) by DeVore, Leviatan and Yu [1] by a delicate technique. In general case, if we allow the function \(f\) to have finitely many sign changes, then the approximation by rational functions with prescribed numerator degree is related to the times of sign change of the function \(f\). This kind of result was first given by Leviatan, Lubinski in [3]. Actually, they proved the following

**Theorem A.** Let \(f(x) \in C_{[-1,1]}\) changes sign exactly \(l\) times in \((-1,1)\), say at \(b_1, b_2, \cdots, b_l\), then for each \(n \geq 1\), there exists a polynomial \(p_n(x) \in \Pi_n\), having the same sign as \(f\) in \((b_1, 1)\), and such that for \(x \in [-1, 1]\),
\[
\left\| f(x) - \frac{\prod_{j=1}^{l} (x - b_j)}{p_n(x)} \right\|_{C} \leq C(l + 1)^2 \omega(f, 1/n)_{C},
\]
where \(\| \cdot \|_{C}\) and \(\omega(f, t)_{C}\) stand for the uniform norm and the modulus of continuity of \(f\) in the continuous function space.

If the function \(f\) changes sign \(l\) times in \((-1,1)\), a natural problem is that whether the Jackson type estimation still holds for approximation by rational functions with prescribed numerator degree in \(L^p\) norm for \(1 \leq p < \infty\)? The present paper will give it a positive answer for \(1 < p < \infty\) by a different approach from [3] (a partial case \(k = 1\) was achieved in [6]).

**Definition.** Let \(f(x) \in L^p_{[-1,1]}\). If there are \(l\) points \(-1 < a_1 < a_2 < \cdots < a_l < 1\) such that
\[
\sigma \text{sgn}(\prod_{j=1}^{l} (x - a_j)) f(x) \geq 0, \ x \in [-1, 1], \ \sigma = \pm 1,
\]
and for every \(j = 1, 2, \cdots, l\) and any \(0 < \eta < a_{j+1} - a_j\ (a_{l+1} = 1)\),
\[
\text{mes}\left(\{x \in (a_j, a_{j+1}) : f(x) \neq 0\} \cap (a_j, a_j + \eta)\right) > 0,
\]

where \(\text{mes}\) is the measure of a set in the real line.
then we say $f(x)$ changes sign exactly $l$ times at $a_1, a_2, \ldots, a_l$.

**Theorem.** Let $l$ be a natural number, $1 < p < \infty$. If $f(x) \in L^p_{[-1,1]}$ changes sign exactly $l$ times in $(-1,1)$, then there exist a $r(x) \in \mathbb{R}_n[-1,1]$ such that

$$
\|f(x) - r(x)\|_{L^p} \leq C_{p,l,b}(f, n^{-1})_{L^p},
$$

where $C_{p,l,b}$ is a positive constant only depending on $p$, $l$ and $b$ which relates to the distance among the sign change points of $f(x)$ and will be given in $\S 3$.

Here and throughout the whole paper, we always use $C$ to indicate a absolute positive constant, and $C_x$ to indicate a positive constant only depending upon $x$, their values may vary in different occurrences even in the same line.

## §2. Preliminaries

We need several lemmas first.

**Lemma 1([8, Lemma 3]).** Let $f(x) \in L^p_{[-1,1]}, 1 \leq p \leq \infty$. Extend $f(x)$ as follows:

$$
F_n(x) = \begin{cases} 
  n \int_0^{1/n} f(t) \, dt, & x \in [-2, -1), \\
  f(x), & x \in [-1, 1], \\
  n \int_{1-1/n}^{1} f(t) \, dt, & x \in (1, 2].
\end{cases}
$$

Then $F_n(x) \in L^p_{[-2,2]}$ and

$$
\omega(F_n, n^{-1})_{L^p_{[-2,2]}} \leq C\omega(f, n^{-1})_{L^p}.
$$

**Lemma 2.** Let $f(x) \in L^p_{[-1,1]}, 1 \leq p \leq \infty$, changes sign exactly $l$ times in $(-1,1)$, write

$$
f_h(x) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(u) \, du
$$

as the Steklov function of $f(x)$, then for sufficiently small $h > 0$, $f_h(x)$ also change sign exact $l$ times in $(-1 + h/2, 1 - h/2)$.

**Proof.** Let $a_1 < a_2 \cdots < a_l$ be the $l$ sign change points of $f(x)$ in $(-1,1)$, write $a_0 = -1$, $a_{l+1} = 1$ and $a = \min\{a_j - a_{j-1} : j = 1, 2, \ldots, l + 1\}$. Fix $h < a/3$. We will prove that $f_h(x)$ changes sign once at every small interval $(a_j - h/2, a_j + h/2)$, $j = 1, 2, \cdots, l$, while keeps sign in $[-1,1] \setminus \bigcup_{j=1}^l (a_j - h/2, a_j + h/2)$.

Without loss of generality, assume that $f(x) \leq 0$ for $x \in (a_{j-1}, a_j)$, and $f(x) \geq 0$ for $(a_j, a_{j+1})$. In view of that (see the Definition)

$$
f_h(a_{j-1} + h/2) = \frac{1}{h} \int_{a_{j-1}}^{a_{j-1}+h} f(u) \, du < 0,
$$
we see that there is at least a point \( b_j \in (a_{j-1} + h/2, a_{j+1} - h/2) \) at which \( f(x) \) changes sign. In addition, \( f_h(b_j) = 0 \). For any \( x \in [a_{j-1} + h/2, a_j - h/2] \), we have

\[
x - h/2 \geq a_{j-1} \quad \text{and} \quad x + h/2 \leq a_j,
\]

so that \( f_h(x) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(u) du \leq 0 \), thus we see that \( f_h(x) \) keeps sign in \((a_{j-1} + h/2, a_j - h/2)\). Similarly, \( f_h(x) \) keeps sign in \((a_j + h/2, a_{j+1} - h/2)\). That means, \( b_j \in (a_j - h/2, a_j + h/2) \). We still need to show that \( b_j \) is unique. In fact,

\[
\int_{b_j-h/2}^{b_j+h/2} f(u) du = 0,
\]

let \( x \in (b_j, a_j + h/2) \), with noticing that \( a_{j-1} \leq a_j - h \leq b_j - h/2 \leq x - h/2 \leq a_j \), and \( a_j \leq b_j + h/2 \leq x + h/2 \leq a_j + h \leq a_{j+1} \), we deduce that

\[
\int_{x-h/2}^{x+h/2} f(u) du = \int_{b_j-h/2}^{b_j+h/2} f(u) du + \int_{b_j+h/2}^{x+h/2} f(u) du
\]

\[
\geq \int_{b_j-h/2}^{b_j+h/2} f(u) du + \int_{b_j+h/2}^{x+h/2} f(u) du \geq 0.
\]

Similarly, for \( x \in (a_j - h/2, b_j) \), \( f_h(x) \) keeps a negative sign. Thus we have proved Lemma 2.

It is easy to prove the following lemma.

**Lemma 3.** Let \( f(x) \in L^p_{[-1,1]}, 1 \leq p \leq \infty \), \( f_h(x) \) be the Steklov function of \( f(x) \), write

\[
f_{hh}(x) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f_h(u) du
\]

as the second order Steklov function of \( f(x) \), then for sufficiently small \( h > 0 \), we have

\[
\| f(x) - f_h(x) \|_{L^p_{[-1+h/2,1-h/2]}} \leq C \omega(f,h)_{L^p},
\]

\[
\| f(x) - f_{hh}(x) \|_{L^p_{[-1+h,1-h]}} \leq C \omega(f,h)_{L^p},
\]

\[
\| f_{hh}(x) \|_{L^p_{[-1+h,1-h]}} \leq C h^{-1} \omega(f,h)_{L^p},
\]

\[
\| f^{(n)}_{hh}(x) \|_{L^p_{[-1+h,1-h]}} \leq C h^{-n} \omega(f,h)_{L^p}.
\]

**Lemma 4([1, Lemma 3.2]).** Write

\[
\lambda_n(t) = c_n \left\{ \left( \frac{\sin n(t-\delta_n)/2}{\sin(t-\delta_n)/2} \right)^4 + \left( \frac{\sin n(t+\delta_n)/2}{\sin(t+\delta_n)/2} \right)^4 \right\}.
\]
where \( \delta_n = \frac{1}{2n} \) and \( c_n \) is taken such that \( \int_{-\pi}^{\pi} \lambda_n(t) \, dt = 1 \). Let \( f(x) \) be a \( p \) power integrable function with period \( 2\pi \) (in symbol \( f(x) \in L^p_{2\pi} \)), define \( \Lambda_n(f, x) = \int_{-\pi}^{\pi} f(x + s) \lambda_n(s) \, ds \), then for \( 1 \leq p < \infty \), we have

\[
\begin{align*}
\|f - \Lambda_n(f)\|_{L^p_{2\pi}} &\leq C \omega(f, 1/n)_{L^p_{2\pi}}, \\
\omega(\Lambda_n(f), t)_{L^p_{2\pi}} &\leq C \omega(f, t)_{L^p_{2\pi}}, \\
\sup_{-\pi \leq x \leq \pi} \Lambda_n(f, x) &\leq C(1 + n|t|)^4, \\
\int_{-\pi}^\pi \varphi \lambda_n(t) \, dt &\sim n^{-j}, \quad j = 0, 1, 2.
\end{align*}
\]

**Lemma 5**([7, p.5]). Let \( f(x) \in L^p_{[a, b]}, 1 < p \leq \infty, x \in I \subset [a, b] \), define

\[
M(f, x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(u)| \, du,
\]

then

\[
\|M(f, x)\|_{L^p_{[a, b]}} \leq C_p \|f\|_{L^p_{[a, b]}}.
\]

The following Lemma 6 is a fundamental inequality of own independent interest.

**Lemma 6.** Let \( l \geq 1 \), \( d_1 < d_2 < \cdots < d_l \), \( d = \min\{d_{j+1} - d_j : j = 1, 2, \ldots, l-1\} \). Then for given numbers \( A \) and \( B, B \neq d_j \), it holds that

\[
|J_l| = \left| 1 - \frac{\prod_{j=1}^{l}(A - d_j)}{\prod_{j=1}^{l}(B - d_j)} \right| \leq C_{d, l} M_l \sum_{j=1}^{l} \frac{1}{|B - d_j|},
\]

where

\[
M_l = \left\{ \begin{array}{ll}
|B - A|, & |B - A| \leq 1, \\
\end{array} \right.
\]

**Proof.** We will prove Lemma 6 by induction. For \( l = 1 \), it is clear that

\[
J_1 = 1 - \frac{A - d_1}{B - d_1} = \frac{B - A}{B - d_1},
\]

we then clearly have the required inequality (1). Assume (1) holds for \( l = m \), that is,

\[
|J_m| \leq C_{d, m} M_m \sum_{j=1}^{m} \frac{1}{|B - d_j|}.
\]

Then for \( l = m + 1 \), we check that

\[
J_{m+1} = 1 - \frac{\prod_{j=1}^{m+1}(A - d_j)}{\prod_{j=1}^{m+1}(B - d_j)}
= \frac{B - A}{B - d_{m+1}} + \left( 1 - \frac{\prod_{j=1}^{m}(A - d_j)}{\prod_{j=1}^{m}(B - d_j)} \right) \frac{A - d_{m+1}}{B - d_{m+1}}.
\]
Thus by the assumption,

\[
|J_{m+1}| \leq \frac{|B - A|}{|B - d_{m+1}|} + C_{d,m} M_m \sum_{j=1}^{m} \frac{1}{|B - d_j|} \left( 1 + \frac{|B - A|}{|B - d_{m+1}|} \right)
\]

\[
\leq \frac{|B - A|}{|B - d_{m+1}|} + C_{d,m} M_m \sum_{j=1}^{m} \frac{1}{|B - d_j|}
\]

\[
+ C_{d,m} M_m |B - A| \sum_{j=1}^{m} \frac{1}{|B - d_j|} \frac{1}{|B - d_{m+1}|}
\]

\[
\leq C_{d,m+1} M_{m+1} \sum_{j=1}^{m+1} \frac{1}{|B - d_j|}.
\]

In the last step above, an inequality

\[
\frac{1}{|y - x||z - x|} \leq \frac{1}{|y - x|} \left( \frac{1}{|y - x|} + \frac{1}{|z - x|} \right)
\]

is applied. By induction, Lemma 6 is proved.

For our purpose, applying \( A = \cos \theta, B = \cos(\theta + s) \) and \( d_j = b_j \), we get

**Lemma 7.** Let \( l \geq 1, -1 < b_1 < b_2 < \cdots < b_l < 1, b = \min\{b_{j+1} - b_j : j = 1, 2, \ldots, l - 1\} \). Then for \( \theta, s \in [-\pi, \pi] \) and \( \cos(\theta + s) \neq b_j, j = 1, 2, \ldots, l \), it holds that

\[
\left| 1 - \frac{\prod_{j=1}^{l}(\cos \theta - b_j)}{\prod_{j=1}^{l}(\cos(\theta + s) - b_j)} \right| \leq C_{b,l} \sum_{j=1}^{l} \frac{|s|}{|\cos(\theta + s) - b_j|}.
\]

**§3. Proof of the Theorem**

We always assume \( 1 < p < \infty \). Suppose \( f(x) \in L_p([-1,1]) \) changes sign exactly \( l \) times in \((-1,1)\), \( f \neq 0 \), extend \( f(x) \in L_p([-1,1]) \) to \( F_n(x) \in \hat{L}_p([-2,2]) \) in a way we described in Lemma 1. Obviously \( F_n(x) \) changes sign exactly \( l \) times in \((-2,2)\) and satisfies

\[
\omega(F_n, n^{-1})_{L_p([-2,2])} \leq C \omega(f, n^{-1})_{L_p}.
\]

Take a sufficiently small \( h > 0 \), for \( x \in [-2,2] \), we define the second order Steklov function \( \tilde{F}_n(x) \) for \( F_n(x) \), that is, \( \tilde{F}_n(x) = (F_n(x))_{hh} \) (see the definition of Lemma 3). Corresponding to the \( l \) sign change points \( a_1 < a_2, \cdots < a_l \) of \( f(x) \) in \((-1,1)\), from Lemma 2 we have \( l \) points \(-1 < b_1 < b_2 < \cdots < b_l < 1 \) (note \( F_n(x) \) equals constants outside \([-1,1]\) at which \( \tilde{F}_n(x) \) changes sign. Setting \( b = \min\{b_{j+1} - b_j : j = 1, 2, \cdots, l - 1\} \), this is the factor we mentioned in the statement of the Theorem.
Assume \( \text{sgn} \left( \prod_{j=1}^{l} (x - b_j) \right) \hat{F}_n(x) \geq 0 \). For given an \( \epsilon > 0 \) and \( x \in [-1, 1]([-1, 1] \subseteq [-2 + h, 2 - h]) \), set

\[
g(x) = \frac{\hat{F}_n(x)}{\prod_{j=1}^{l}(x - b_j)} + \epsilon, \quad G(\theta) = g(\cos \theta),
\]

\[
\bar{g}(x) = \Lambda_n(G, \theta) = \int_{-\pi}^{\pi} g(\cos(\theta + s)) \lambda_n(s) \, ds.
\]

Let

\[
K_n(t) = d_n \left( \frac{\sin nt/2}{\sin t/2} \right)^8,
\]

where \( d_n \) is the normalizing constant satisfying \( \int_{-\pi}^{\pi} K_n(t) \, dt = 1 \). It is easy to see (see [5]) \( d_n \sim n^{-7} \) and

\[
\int_{-\pi}^{\pi} |t|^j K_n(t) \, dt \sim n^{-j}, \quad j = 0, 1, \ldots, 6.
\]

Define

\[
P_n(x) = L_n \left( \frac{1}{\bar{g}(x)} \right) = \int_{-\pi}^{\pi} \frac{1}{\Lambda_n(G, \theta + t)} K_n(t) \, dt.
\]

By \( g(x) > 0 \), \( \bar{g}(x) > 0 \). So the definition of \( P_n(x) \) is reasonable and is a polynomial of degree \( n \). We always take \( \epsilon = \omega(f, n^{-1})_{L^p}, h = n^{-1} \). Note that

\[
\left\| f(x) - \prod_{j=1}^{l}(x - b_j) \right\|_{L^p}
\]

\[
\leq \left\| f(x) - \prod_{j=1}^{l}(x - b_j)g(x) \right\|_{L^p} + \left\| \prod_{j=1}^{l}(x - b_j)(g(x) - \bar{g}(x)) \right\|_{L^p}
\]

\[
+ \left\| \prod_{j=1}^{l}(x - b_j) \left( \bar{g}(x) - \frac{1}{P_n(x)} \right) \right\|_{L^p}
\]

= : \left\| I_1 \right\|_{L^p} + \left\| I_2 \right\|_{L^p} + \left\| I_3 \right\|_{L^p}.
\]

By Lemma 1 and Lemma 3, we obtain that

\[
\left\| I_1 \right\|_{L^p} = \left\| f(x) - \hat{F}_n(x) - \epsilon \prod_{j=1}^{l}(x - b_j) \right\|_{L^p}
\]

\[
\leq \left\| F_n(x) - \hat{F}_n(x) - \epsilon \prod_{j=1}^{l}(x - b_j) \right\|_{L^p_{[-2, 2]}}\]

\[
\leq C \left( \omega(F_n, h)_{L^p_{[-2, 2]}} + \epsilon \right)
\]

\[\leq C \omega(f, n^{-1})_{L^p}. \quad (2)\]
At the same time,
\[
\| I_2 \|_{L^p} = \left\| \prod_{j=1}^{l} (x - b_j) \frac{g(x) - \hat{g}(x)}{\lambda_n(s)} \right\|_{L^p}
\]
\[
= \left\| \int_{-\pi}^{\pi} \left( \frac{F_n(\cos \theta) - \sum_{j=1}^{l} (\cos \theta - b_j)}{\prod_{j=1}^{l} (\cos(\theta + s) - b_j)} \hat{F}_n(\cos(\theta + s)) \right) \lambda_n(s) \, ds \right\|_{L^p}
\]
\[
\leq \left\| \int_{-\pi}^{\pi} \left( \frac{F_n(\cos \theta) - \hat{F}_n(\cos(\theta + s))}{\prod_{j=1}^{l} (\cos(\theta + s) - b_j)} \right) \lambda_n(s) \, ds \right\|_{L^p}
\]
\[
+ \left\| \int_{-\pi}^{\pi} \left( 1 - \frac{\prod_{j=1}^{l} (\cos \theta - b_j)}{\prod_{j=1}^{l} (\cos(\theta + s) - b_j)} \right) \hat{F}_n(\cos(\theta + s)) \lambda_n(s) \, ds \right\|_{L^p}
\]
\[
=: \| I_{21} \|_{L^p} + \| I_{22} \|_{L^p}.
\]
By Lemma 4 and Lemma 5, then
\[
| I_{21} | = \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\cos(\theta + s)} \frac{\hat{F}_n'(u) du}{\lambda_n(s)} \right|
\]
\[
\leq \int_{-\pi}^{\pi} \frac{1}{\cos(\theta + s) - \cos \theta} \int_{-\pi}^{\cos(\theta + s)} |\hat{F}_n'(u)| du \left| \cos(\theta + s) - \cos \theta \right| \lambda_n(s) \, ds
\]
\[
\leq CM(\hat{F}_n, x) \int_{-\pi}^{\pi} |s| \lambda_n(s) \, ds \leq Cn^{-1} M(\hat{F}_n, x),
\]
while applying Lemma 7 we get
\[
| I_{22} | \leq C_{b, l} \sum_{j=1}^{l} \int_{-\pi}^{\pi} \frac{\hat{F}_n(\cos(\theta + s))}{\cos(\theta + s) - b_j} \left| s \right| \lambda_n(s) \, ds.
\]  

In order to estimate \( | I_{22} | \), we consider two cases.

**Case 1. For some \( j \in \{1, 2, \ldots, l\} \) and \( s \in [-\pi, \pi], \ |\cos(\theta + s) - b_j| \leq h. \)

By noticing that \( \hat{F}_n(b_j) = 0 \), we have
\[
\int_{-\pi}^{\pi} \frac{\hat{F}_n(\cos(\theta + s))}{\cos(\theta + s) - b_j} \left| s \right| \lambda_n(s) \, ds
\]
\[
\leq \int_{-\pi}^{\pi} \frac{1}{\cos(\theta + s) - b_j} \int_{b_j}^{\cos(\theta + s)} \left( \hat{F}_n'(u) - \hat{F}_n'(x) \right) \left| s \right| \lambda_n(s) \, ds + \int_{-\pi}^{\pi} \| \hat{F}_n'(x) \|_{L^p} \left| s \right| \lambda_n(s) \, ds.
\]
For any \( x, u \in [-1, 1] \), by noting \( x = \cos \theta \) and that \( u \) lies between \( \cos(\theta + s) \) and \( b_j \), we obtain that
\[
| \hat{F}_n'(x) - \hat{F}_n'(u) | = \left| \frac{1}{x - u} \int_{u}^{x} \hat{F}_n''(y) dy \right| \left| x - u \right|
\]
\[
\leq \left| \frac{1}{x - u} \int_{u}^{x} \hat{F}_n''(y) \left( |x - \cos(\theta + s)| + |\cos(\theta + s) - b_j| \right) dy \right|
\]
\[
\leq CM(\hat{F}_n^\prime, x) (|s| + h),
\]
therefore by Lemma 4 it follows that

$$\int_{-\pi}^{\pi} \left| \frac{\hat{F}_n(\cos(\theta + s))}{\cos(\theta + s) - b_j} \right| |s| \Lambda_n(s) \, ds \leq C \left( n^{-1} |\hat{F}_n'(x)| + n^{-2} M(\hat{F}_n''(x)) \right).$$

**Case 2. For some** $j \in \{1, 2, \cdots, l\}$ **and** $s \in [-\pi, \pi]$, $|\cos(\theta + s) - b_j| > h$.

Again, $\hat{F}_n(b_j) = 0$, by Lemma 5,

$$\left| \frac{\hat{F}_n(\cos(\theta + s))}{\cos(\theta + s) - b_j} - \frac{\hat{F}_n(\cos \theta)}{\cos \theta - b_j} \right| = \left| \frac{(\cos \theta - b_j) \hat{F}_n(\cos(\theta + s)) - (\cos(\theta + s) - b_j) \hat{F}_n(\cos \theta)}{(\cos(\theta + s) - b_j)(\cos \theta - b_j)} \right|$$

$$\leq \frac{1}{|\cos(\theta + s) - b_j|} \left| \hat{F}_n(\cos(\theta + s)) - \hat{F}_n(\cos \theta) \right| + \frac{-\cos \theta - \cos(\theta + s)}{(\cos(\theta + s) - b_j)(\cos \theta - b_j)} \hat{F}_n(\cos \theta)$$

$$\leq C \left( \frac{|s|}{h} M(\hat{F}_n''(x)) + \frac{|s|}{h} M(\hat{F}_n'(x)) \right) \leq C \left( \frac{|s|}{h} M(\hat{F}_n''(x)) \right).$$

Applying Lemma 5 again, with $h = n^{-1}$ we get

$$\left| \frac{\hat{F}_n(\cos(\theta + s))}{\cos(\theta + s) - b_j} \right| \leq C \left( \frac{|s|}{h} M(\hat{F}_n''(x)) + \frac{\hat{F}_n(\cos \theta)}{\cos \theta - x_0} \right)$$

$$\leq C \left( 1 + \frac{|s|}{h} \right) M(\hat{F}_n'(x)),$$

hence

$$\int_{-\pi}^{\pi} \left| \frac{\hat{F}_n(\cos(\theta + s))}{\cos(\theta + s) - b_j} \right| |s| \Lambda_n(s) \, ds \leq C n^{-1} M(\hat{F}_n''(x)).$$

Altogether, with (3), at any case, for all $x \in [-1, 1]$ it holds that

$$|I_{22}| \leq C_{b, l} \left( n^{-1} M(\hat{F}_n''(x)) + n^{-1} |\hat{F}_n'(x)| + n^{-2} M(\hat{F}_n''(x)) \right).$$

Applying Lemma 1, Lemma 3 and Lemma 5, we get

$$\|I_2\|_{L^p} \leq \|I_2\|_{L^p} \leq \|I_{21}\|_{L^p} + \|I_{22}\|_{L^p} \leq C \left( n^{-1} \|\hat{F}_n''(x)\|_{L^p} + M(\hat{F}_n'(x)) \right) + n^{-2} \|M(\hat{F}_n''(x))\|_{L^p}$$

$$\leq C_{p} \left( n^{-1} \|\hat{F}_n''(x)\|_{L^p} + n^{-2} \|\hat{F}_n'(x)\|_{L^p} \right) \leq C_{p} \left( n^{-1} h^{-1} \omega(F_n, h)_{L_{-2, \eta}^p} + n^{-2} h^{-2} \omega(F_n, h)_{L_{-2, \eta}^p} \right) \leq C_{p} \omega(f, n^{-1})_{L^p}. \quad (4)$$
To estimate $\|I_3\|_{L^p}$, we consider two sets,

$$E_1 = \left\{ x \in [-1, 1] : \frac{1}{p_n(x)} \geq \tilde{g}(x) \right\}, \quad E_2 = [0, 1] \setminus E_1.$$

For $x \in E_1$, from Cauchy-Schwarz inequality, we see that

$$L_n(\tilde{g}, x)L_n\left(\frac{1}{\tilde{g}}, x\right) \geq L_n^2(1, x) = 1,$$

so that

$$L_n(\tilde{g}, x) \geq \frac{1}{L_n\left(\frac{1}{\tilde{g}}, x\right)} = \frac{1}{p_n(x)}.$$

Hence, for $x \in E_1$, we have

$$|I_3| = \left| \prod_{j=1}^{l} (x - b_j) \left( \tilde{g}(x) - \frac{1}{p_n(x)} \right) \right| \leq \left| \prod_{j=1}^{l} (x - b_j) (\tilde{g}(x) - L_n(\tilde{g}, x)) \right|$$

$$= \left| \prod_{j=1}^{l} (x - b_j) \int_{-\pi}^{\pi} (\Lambda_n(G, \theta) - \Lambda_n(G, \theta + t)) K_n(t)dt \right|$$

$$\leq \int_{-\pi}^{\pi} \left| \prod_{j=1}^{l} (x - b_j) (\Lambda_n(G, \theta) - \Lambda_n(G, \theta + t)) \right| K_n(t)dt,$$

while

$$\left| \prod_{j=1}^{l} (x - b_j) (\Lambda_n(G, \theta) - \Lambda_n(G, \theta + t)) \right|$$

$$= \left| \int_{-\pi}^{\pi} \left( \frac{\prod_{j=1}^{l} (\cos \theta - b_j)}{\prod_{j=1}^{l} (\cos(\theta + t) - b_j)} \hat{F}_n(\cos(\theta + s)) \right) \Lambda_n(s)ds \right|$$

$$\leq \left| \int_{-\pi}^{\pi} \left( \hat{F}_n(\cos(\theta + s)) - \hat{F}_n(\cos \theta) \right) \Lambda_n(s)ds \right|$$

$$+ \left| \int_{-\pi}^{\pi} \left( 1 - \frac{\prod_{j=1}^{l} (\cos \theta - b_j)}{\prod_{j=1}^{l} (\cos(\theta + t) - b_j)} \right) \hat{F}_n(\cos(\theta + s)) \Lambda_n(s)ds \right|$$

$$+ \left| \int_{-\pi}^{\pi} \left( \hat{F}_n(\cos(\theta + t + s)) - \hat{F}_n(\cos \theta) \right) \Lambda_n(s)ds \right|$$

$$+ \left| \int_{-\pi}^{\pi} \left( 1 - \frac{\prod_{j=1}^{l} (\cos \theta - b_j)}{\prod_{j=1}^{l} (\cos(\theta + t + s) - b_j)} \right) \hat{F}_n(\cos(\theta + t + s)) \Lambda_n(s)ds \right|$$

$$=: |I_{31}| + |I_{32}| + |I_{33}| + |I_{34}|.$$

For $|I_{31}|$ and $|I_{33}|$, noting that $x = \cos \theta$, by Lemma 4 and Lemma 5, we get

$$|I_{31}| \leq Cn^{-1}M(\hat{F}_n', x),$$

$$|I_{33}| \leq C(n^{-1} + |t|)M(\hat{F}_n', x),$$

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\begin{equation}
|I_{32}| = |I_{22}| \leq C_{b,t} \left( n^{-1} M(\hat{F}_n', x) + n^{-1} |\hat{F}'_n(x)| + n^{-2} M(\hat{F}''_n, x) \right).
\end{equation}

For $|I_{34}|$, similar to the estimation of $|I_{22}|$, for given $s, t \in [-\pi, \pi]$, if $|\cos(\theta + s + t) - b_j| \leq h$, we obtain
\begin{equation}
\frac{|\hat{F}_n(\cos(\theta + s + t))|}{\cos(\theta + s + t) - b_j} \leq \left( M(\hat{F}''_n, x)(|t| + |s| + h) + |\hat{F}'_n(x)| \right),
\end{equation}
for given $s, t \in [-\pi, \pi]$, if $|\cos(\theta + s + t) - b_j| > h$, we get
\begin{equation}
\frac{|\hat{F}_n(\cos(\theta + s + t))|}{\cos(\theta + s + t) - b_j} \leq C M(\hat{F}_n', x)(1 + h^{-1}(|t| + |s|)).
\end{equation}
Therefore for $x \in [-1, 1]$, we have
\begin{equation}
|I_{34}| \leq C_{b,t} \sum_{j=1}^{l} \int_{-\pi}^{\pi} \left| \hat{F}_n(\cos(\theta + s + t)) \right| |s| \lambda_n(s) ds
\leq C_{b,t} \left( n^{-1} |\hat{F}_n'(x)| + (n^{-1} + |t|) M(\hat{F}_n', x) + (n^{-2} + n^{-1}|t|) M(\hat{F}''_n, x) \right).
\end{equation}

For all $x \in E_1$, by
\begin{equation}
\int_{-\pi}^{\pi} |t|^j K_n(t) dt \sim n^{-j}, \quad j = 0, 1, \ldots, 6,
\end{equation}
we obtain
\begin{equation}
|I_3| \leq \int_{-\pi}^{\pi} (|I_{31}| + |I_{32}| + |I_{33}| + |I_{34}|) K_n(t) dt
\leq C_{b,t} \left( n^{-1} |\hat{F}_n'(x)| + n^{-1} M(\hat{F}_n', x) + n^{-2} M(\hat{F}''_n, x) \right).
\end{equation}

thus
\begin{equation}
||I_3||_{L^p_E} \leq ||I_3||_{L^p}
\leq C_{b,t} \left( n^{-1} ||\hat{F}_n'||_{L^p} + n^{-1} ||M(\hat{F}_n)||_{L^p} + n^{-2} ||M(\hat{F}''_n)||_{L^p} \right)
\leq C_{p,b,i} \omega(F_n, n^{-1})_{L^p_{[-2,2]}} \leq C_{p,b,i} \omega(f, n^{-1})_{L^p}.
\end{equation}

On the other hand, for $x \in E_2$, the inequality $\frac{1}{P_n(x)} \leq \bar{g}(x)$ holds. Noticing that
\begin{equation}
\bar{g}(x) = \Lambda_n(G, \theta), \quad P_n(x) = \int_{-\pi}^{\pi} \frac{1}{\Lambda_n(G, \theta + t)} K_n(t) dt,
\end{equation}
and applying Lemma 4, we get
\begin{equation}
0 \leq \bar{g}(x) - \frac{1}{P_n(x)} = \int_{-\pi}^{\pi} \frac{\Lambda_n(G, \theta + t) - \Lambda_n(G, \theta))}{\Lambda_n(G, \theta + t) P_n(x)} K_n(t) dt
\leq \int_{-\pi}^{\pi} (\Lambda_n(G, \theta + t) - \Lambda_n(G, \theta)) \frac{\Lambda_n(G, \theta)}{\Lambda_n(G, \theta + t)} K_n(t) dt
\leq \int_{-\pi}^{\pi} (\Lambda_n(G, \theta + t) - \Lambda_n(G, \theta)) (1 + n|t|)^4 K_n(t) dt.
\end{equation}
Write \( \tilde{K}_n(t) = (1 + n|t|)^4 K_n(t) \), then for all \( x \in E_2 \), we have

\[
|\tilde{I}_3| = \left| \prod_{j=1}^{l} (x - b_j) \left( \tilde{g}(x) - \frac{1}{p_n(x)} \right) \right| \\
\leq \int_{-\pi}^{\pi} \left| \prod_{j=1}^{l} (x - b_j) \left( \Lambda_n(G, \theta) - \Lambda_n(G, \theta + t) \right) \right| \tilde{K}_n(t) \, dt \\
\leq \int_{-\pi}^{\pi} \left( \tilde{F}_n(\cos(\theta + s)) - \tilde{F}_n(\cos(\theta)) \right) \lambda_n(s) \, ds \left| \tilde{K}_n(t) \right| dt \\
+ \int_{-\pi}^{\pi} \left( 1 - \frac{\prod_{j=1}^{l} (\cos(\theta - b_j))}{\prod_{j=1}^{l} (\cos(\theta + s) - b_j)} \right) \tilde{F}_n(\cos(\theta + s)) \lambda_n(s) \, ds \left| \tilde{K}_n(t) \right| dt \\
+ \int_{-\pi}^{\pi} \left( \tilde{F}_n(\cos(\theta + t + s)) - \tilde{F}_n(\cos(\theta)) \right) \lambda_n(s) \, ds \left| \tilde{K}_n(t) \right| dt \\
+ \int_{-\pi}^{\pi} \left( 1 - \frac{\prod_{j=1}^{l} (\cos(\theta - b_j))}{\prod_{j=1}^{l} (\cos(\theta + t + s) - b_j)} \right) \tilde{F}_n(\cos(\theta + t + s)) \lambda_n(s) \, ds \left| \tilde{K}_n(t) \right| dt \\
=: \tilde{I}_{31} + \tilde{I}_{32} + \tilde{I}_{33} + \tilde{I}_{34}.
\]

From the property of \( K_n(t) \), we have

\[
\int_{-\pi}^{\pi} K_n(\tilde{t}) \, dt \leq C, \quad \int_{-\pi}^{\pi} |t|^j K_n(\tilde{t}) \, dt \sim n^{-j}, \quad j = 0, 1, 2.
\]

Similar to the estimation of \(|I_{3j}| (j = 1, 2, 3, 4)\), we obtain that

\[
|\tilde{I}_{31}| \leq \int_{-\pi}^{\pi} \left| I_{31} \right| \tilde{K}_n(t) \, dt \leq C_{b, l} n^{-1} M(\tilde{F}_n', x),
\]

\[
|\tilde{I}_{32}| \leq \int_{-\pi}^{\pi} \left| I_{32} \right| \tilde{K}_n(t) \, dt \leq C_{b, l} \left( n^{-1} M(\tilde{F}_n', x) + n^{-1} |\tilde{F}_n'(x)| + n^{-2} M(\tilde{F}_n'', x) \right),
\]

\[
|\tilde{I}_{33}| \leq \int_{-\pi}^{\pi} \left| I_{33} \right| \tilde{K}_n(t) \, dt \leq C_{b, l} n^{-1} M(\tilde{F}_n', x),
\]

\[
|\tilde{I}_{34}| \leq \int_{-\pi}^{\pi} \left| I_{34} \right| \tilde{K}_n(t) \, dt \leq C_{b, l} \left( n^{-1} |\tilde{F}_n'(x)| + n^{-1} M(\tilde{F}_n', x) + n^{-2} M(\tilde{F}_n'', x) \right).
\]

Hence for all \( x \in E_2 \), the inequality

\[
|I_3| \leq C_{b, l} \left( n^{-1} |\tilde{F}_n'(x)| + n^{-1} M(\tilde{F}_n', x) + n^{-2} M(\tilde{F}_n'', x) \right)
\]

holds, so it follows that

\[
\| I_3 \|_{L^2_{x \in E_2}} \leq \| I_3 \|_{L^p} \leq C_{b, l} \left( n^{-1} \| F_n' \|_{L^p} + n^{-1} \| M(\tilde{F}_n') \|_{L^p} + n^{-2} \| M(\tilde{F}_n'') \|_{L^p} \right) \leq C_{p, b, l} \omega(F_n, n^{-1})_{L^p_{[-2, 2]}} \leq C_{p, b, l} \omega(f, n^{-1})_{L^p}.
\]
In other words, for all \( x \in [-1, 1] \), we have
\[
\|I_3\|_{L^p} \leq C_{p,b,\omega}(f, n^{-1})_{L^p}.
\] (5)

Finally, by (2), (4) and (5), we get
\[
\left\| f(x) - \frac{\Pi_{j=1}^n (x - b_j)}{P_n(x)} \right\|_{L^p} \leq C_{p,b,\omega}(f, n^{-1})_{L^p}.
\]

Thus we are done.

References


