Coxeter Groups and the Moussong Complex.

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Abstract

For any Coxeter Group, we can construct a CAT(0) metric cell complex, on which the group acts co-compactly discretely by isometries. This theorem is due to Moussong, hence the complex is called the Moussong Complex. To understand the theorem, we need to consider some geometry, and find some equivalent characterisations of a space being CAT(κ) for various values of κ ∈ ℝ. We also need some group theory, in particular to consider some special facts about Coxeter Groups. Putting these together will enable us to construct the complex we want.
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Introduction

A group is a relatively easy concept to define, but its structure and properties can be elusive. For example, a simple way to describe a group is by a presentation, but as shown by Boone and Novikov, the problems of determining whether a word given in terms of a presentation is the identity, or when two words are conjugate, are in general unsolvable! The latter is known as the conjugacy problem, and its solution implies a solution to the former.

One way to attack groups is to re-express them in different forms. If we can turn group theoretic information, which is usually highly abstract and laden with symbols, into a geometric object, we often find the geometry conceptually easier to interpret. The idea of geometric group theory is to relate properties of such geometric objects back to the group from which they are derived.

A Coxeter group is one with a presentation such that each generator is an involution, and each relation is of the form \((s_is_j)^{m_{ij}}\) where \(s_i, s_j\) are generators and \(m_{ij}\) is a natural number or infinity if there is no relation. In this thesis, we will consider the construction of a complex, with cells corresponding to certain subgroups of Coxeter groups. We will give each cell a metric which will be consistent between the cells. We will talk about the curvature of the complex, using a special characterisation of curvature in arbitrary metric spaces, known as the CAT(\(\kappa\))-inequality.

We show that every Coxeter group acts in an appropriate way on a complex that is CAT(0). This is a useful result. For example, there is a theorem by Short and Bridson-Alonso which says that any group which acts appropriately on a CAT(0) space has a solvable conjugacy problem. Therefore our result tells us that Coxeter groups satisfy the conditions for this theorem.

In the first chapter we lay the ground-work by considering various properties and constructions of CAT(\(\kappa\)) spaces. We will see that if a space is locally CAT(\(\kappa\)) and simply connected, then it is CAT(\(\kappa\)) globally. Next we define what we mean by metric complexes, in particular simplicial and cubical ones. We see that each cell of such a complex is isometric to a convex part of model space, which is a space with constant curvature \(\kappa\). Thus when we glue the cells together, we need to know under what conditions the curvature of the cells is preserved in the complex. This will be the Link Condition.

In the last chapter we describe some basic facts of Coxeter groups. We describe the reflection representation, which gives a way for a Coxeter group to act on real \(n\)-space. This gives us a clue to the construction of a Euclidean...
cell complex, where each cell is a convex part of Euclidean space, determined by the action of the group on Euclidean space via the representation.

Finally we construct the Moussong Complex and using our previous work, show it is CAT(0).
Chapter 1

CAT(κ) Spaces

1.1 Geodesic Metric Spaces

A metric on a set $X$ is a function $d : X \times X \to \mathbb{R}$ that satisfies the following conditions:

1. $d(x, y) \geq 0 \ \forall \ x, y \in X$

2. $d(x, y) = 0$ if and only if $x = y$

3. $d(x, y) = d(y, x) \ \forall \ x, y \in X$

4. $d(x, z) \leq d(x, y) + d(y, z) \ \forall \ x, y, z \in X$

Condition 4 is called the triangle inequality. If $d$ satisfies all except condition 2 we say it is a pseudometric. If $d$ is a metric on $X$, we say $(X, d)$ is a metric space, and $d(x, y)$ is the distance between $x$ and $y$ in $X$.

Let $X = (X, d)$ be a metric space. A map $c : [0, a] \to X$ is a geodesic from $x$ to $y$ if $c(0) = x$, $c(a) = y$ and $d(c(t), c(t')) = |t - t'| \ \forall \ t, t' \in [0, a]$. We call the image of the path $c$ in $X$ a geodesic segment from $x$ to $y$, and denote it by $[x, y]$. The path $c$ may not necessarily be unique, so we denote another geodesic segment from $x$ to $y$ by $[x, y]'$, and so on.

If for each pair of points $x, y \in X$ there is a geodesic segment $[x, y]$, then we say $X$ is a geodesic metric space. If for each pair of points $x, y \in X$ of distance less than some positive real number $r$ apart there is a geodesic segment $[x, y]$, then we say $X$ is an $r$-geodesic metric space. If in each case
above, the geodesic segments are unique, we say \( X \) is \textit{uniquely geodesic} or \( r \)-\textit{uniquely geodesic} respectively. Unique geodesics vary continuously with their endpoints.

Finally, a map \( c : [0,a] \to X \) is a \textit{local geodesic} if for each \( t \in [0,a] \) \( \exists \epsilon > 0 \) such that \( c|_{[t-\epsilon,t+\epsilon]} \) is a geodesic.

\section{1.2 Euclidean Space}

Take real \( n \)-space \( \mathbb{R}^n, x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n \). Define the \textit{Euclidean scalar product} on \( \mathbb{R}^n \) by \( (x|y) = \sum_{i=1}^{n} x_i y_i \) and the \textit{Euclidean norm} by \( \|x\| = (x|x)^{1/2} \). Then we can define a function \( d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) by

\[
d(x,y) = \|x - y\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}
\]

which satisfies the conditions for a metric. The triangle inequality follows from the \textit{Cauchy-Schwartz inequality} that \( |(x|y)| \leq \|x\| \|y\| \) (See \cite{10} p.4 for proof). We call the metric space \((\mathbb{R}^n,d)\) \textit{Euclidean n-space}, denoted by \( \mathbb{E}^n \).

Intuitively a geodesic in \( \mathbb{E}^n \), considered as the “shortest path between two points”, would be a straight line in \( \mathbb{R}^n \). This is indeed the case, and moreover such a geodesic \([x,y] = \{ty + (1-t)x : 0 \leq t \leq 1\}\) for each pair of points \( x, y \in \mathbb{E}^n \) is unique. Thus \( \mathbb{E}^n \) is a uniquely geodesic metric space.

Recall the so-called \textit{cosine rule} for a triangle in \( \mathbb{E}^2 \) with geodesic edges of length \( a, b, c \), and vertex angle \( \gamma \):

\[
\begin{align*}
\text{Figure 1.1:} \\
c^2 &= a^2 + b^2 - 2ab\cos\gamma.
\end{align*}
\]

\section{1.3 Comparison Triangles; Angle}

A \textit{geodesic triangle} in a space \( X \) is a triangle whose sides are each geodesics in \( X \). Since \( \mathbb{E}^2 \) is uniquely geodesic, then given three points of specified
distances apart, there are unique geodesics between them forming a geodesic triangle in $\mathbb{E}^2$. We exploit this fact to define a concept of angle that applies to any geodesic metric space.

Let $X$ be a geodesic metric space. A *comparison triangle* in $\mathbb{E}^2$ for a set of points $(p, q, r)$ in $X$ is a geodesic triangle in $\mathbb{E}^2$ with vertices $\bar{p}, \bar{q}, \bar{r}$ such that $d(p, q) = d(\bar{p}, \bar{q})$, $d(q, r) = d(\bar{q}, \bar{r})$, and $d(r, p) = d(\bar{r}, \bar{p})$. Clearly, as mentioned above, such a triangle always exists in $\mathbb{E}^2$ and is unique up to isometry. The interior angle at $\bar{p}$ is called the *comparison angle* between $[p, q]$ and $[p, r]$ at $p$, and is denoted $\bar{\angle} p(q, r)$.

![Comparison Triangle](image1.2)

**Figure 1.2:** $\alpha = \bar{\angle} p(q, r)$

Note that this angle is always $\leq \pi$ to make sense in $\mathbb{E}^2$. Of course, this angle might vary if you compare a smaller triangle within $(p, q, r)$ in $X$, i.e.

![Comparison Triangle](image1.3)

**Figure 1.3:**

In fact, by the cosine rule, we might expect $\bar{\angle} p(x, y)$ to be less than $\bar{\angle} p(q, r)$ if the triangle in $X$ is “skinnier” than in $\mathbb{E}^2$, or the opposite if it is “fatter”.

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So a more useful definition of angle should only depend on the angle at the germs of geodesics from a point, and not on particular triangles.

**Definition 1.3.1** (Alexandrov)

Let $c : [0, a] \rightarrow X$ and $c' : [0, a'] \rightarrow X$ be two geodesic paths with $c(0) = c'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$, take the comparison triangle $\triangle(c(0), c(t), c'(t'))$, with comparison angle $\angle_{c(0)}(c(t), c'(t'))$. Then define the Alexandrov angle (or simply the angle) between the geodesics $c, c'$ to be the number

$$\angle_{c,c'} \equiv \limsup_{t,t' \to 0} \angle_{c(0)}(c(t), c'(t')) = \limsup_{\epsilon \to 0} \sup_{t,t' < \epsilon} \angle_{c(0)}(c(t), c'(t'))$$

Note that $t$ and $t'$ are independently chosen. The angle only depends on the germs of paths at 0, thus if two paths agree close to 0 but diverge at some later point, the Alexandrov angle cannot distinguish them. Along a geodesic segment, the angle between incoming and outgoing paths at any point is always $\pi$. We will make use of the following “triangle inequality” for angles in $\mathbb{E}^2$.

**Proposition 1.3.1** Let $X$ be a (geodesic) metric space, with $c, c', c''$ three geodesics issuing from the same point in $X$. Then

$$\angle_{c',c''} \leq \angle_{c',c} + \angle_{c,c''}.$$ 


We think of $\mathbb{E}^2$ as having no “curvature”, as a Riemmanian manifold at least. That is, compared to the surface of a sphere for example, Euclidean space seems to be “flat”. Let us define two more geodesic metric spaces, one having “positive” and the other “negative” curvature.

### 1.4 Spherical Space

Consider the $n$-sphere $\mathbb{S}^n$ as the set of points $\{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | (x|x) = 1\}$ sitting in $\mathbb{R}^{n+1}$.

A “great circle” on a sphere is one that if you cut along it would divide the sphere into equal hemispheres. That is, it follows an equatorial line.
Technically it is the intersection of the sphere with a 2-dimensional subspace of $\mathbb{R}^{n+1}$, that is, a plane through the origin.

A simple experiment with some string and a balloon will show that the shortest path between two points in $S^2$ is along a great circle.

The Euclidean length along this path is given by the “arc length”, which we know to be given by $l = r\theta$ where $r$ is the radius (so is 1 here) and $\theta$ is the angle at the origin between the line to these points.

This gives us a big clue, if we want to define a metric on $S^n$ that will have geodesics following great arcs.

Let $d(x, y) = \theta$.

By the cosine rule in $\mathbb{E}^n$ for the triangle

$$\|x - y\|^2 = 1^2 + 1^2 - 2(1)(1) \cos \theta = 2(1 - \cos \theta).$$

Thus $\cos \theta = 1 - \frac{1}{2}\|x - y\|^2 = 1 - \frac{1}{2} \sum_{i=1}^{n+1} (x_i - y_i)^2$

$$= 1 - \sum_{i=1}^{i=n+1} \left( \frac{(x_i)^2}{2} - x_i y_i + \frac{(y_i)^2}{2} \right)$$

$$= 1 - \sum_{i=1}^{i=n+1} \left( \frac{(x_i)^2}{2} + \sum_{i=1}^{i=n+1} x_i y_i - \sum_{i=1}^{i=n+1} \frac{(y_i)^2}{2} \right)$$

$$= \sum_{i=1}^{i=n+1} x_i y_i = (x|y)$$
Then define $d(x, y) \in [0, \pi]$ to be the real number such that $\cos(d(x, y)) = (x|y)$.

Then we have that $d(x, y)$ is a metric on $S^n$.

Proof: The first three conditions are trivial. The triangle inequality is a consequence of the so-called spherical cosine rule, an analogue of the Euclidean one.

Consider a triangle made by three great arcs, as in Figure 1.1, in spherical space $S^n$. Then $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$


Thus $S^n$ is a geodesic metric space, where the geodesics are the great arcs. Formally, given $x \in S^n$, a unit vector $u \in E^{n+1}$ such that $(u|x) = 0$, and a number $a \in [0, \pi]$, the map $c : [0, a] \rightarrow S^n$ given by $c(t) = (\cos t)x + (\sin t)u$ satisfies the conditions for a geodesic. Provided $d(x, y) < \pi$, the geodesic segment $[x, y]$ is unique. Thus $S^n$ is $\pi$-uniquely geodesic.

1.5 Hyperbolic Space

We will consider hyperbolic $n$-space $\mathbb{H}^n$ using the so-called hyperboloid model. Take $\mathbb{R}^{n+1}$, and define the Minkowski inner product $<x|y> = -x_{n+1}y_{n+1} + \sum_{i=1}^{n} x_iy_i$ (a symmetric bilinear form). We denote this inner product space by $E^{n,1}$. Then define $\mathbb{H}^n = \{x \in E^{n,1}| <x|x> = 1 \text{ and } x_{n+1} > 0\}$ and define a metric on $\mathbb{H}^n$ to be the number $d(x, y) \geq 0$ such that $\cosh(d(x, y)) = -<x|y>$. Intuitively, we think of this as a “sphere of radius $\sqrt{-1}$ in $E^{n,1}$”, although I admit this isn’t particularly intuitive. However with this description, things are similar to the $S^n$ case.

Geometrically, for $\mathbb{H}^2$ we have Figure 1.6.

Analogously with $S^n$, a geodesic is given by the intersection of a two dimensional subspace of $\mathbb{R}^{n+1}$ with $\mathbb{H}^n$, or formally, given $x \in \mathbb{H}^n$, unit vector $u \in x^\perp \subseteq E^{n,1}$, where $x^\perp$ is the $n$-dimensional vector subspace of $E^{n,1}$ consisting of all vectors $u$ such that $<u|x> = 0$ (thus $<u|u> = -1$ and $<u|x> = 0$), the map $c : \mathbb{R} \rightarrow \mathbb{H}^n$ given by $c(t) = (\cosh t)x + (\sinh t)u$ satisfies the conditions for a geodesic. There exists such a geodesic segment between any two points in $\mathbb{H}^n$, thus $\mathbb{H}^n$ is a uniquely-geodesic metric space. To prove $d$ is a metric, we note that $\forall x, y \in \mathbb{H}^n$, $<x|y> \leq -1$ with equality if and only if $x = y$. The first three conditions follow roughly from this. The triangle inequality is a consequence of the so-called hyperbolic cosine rule.
Consider a (geodesic) triangle as in Figure 1.1, in hyperbolic space. Then
\[ \cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma. \]


1.6 Model Space

The above descriptions for \( \mathbb{E}^n \), \( \mathbb{S}^n \) and \( \mathbb{H}^n \) suggest that there is a “standard”
uniquely (or \( \pi \)-uniquely) geodesic metric space of respectively 0, +1 and −1
“curvature”. What we really want is some kind of standard or “model”
uniquely geodesic space with some kind of constant curvature \( \kappa \) for any real
number \( \kappa \). This can be easily achieved by “scaling the metric” by some
constant in \( \mathbb{S}^n \) or \( \mathbb{H}^n \) as follows:

For \( \kappa > 0 \), \( M^\kappa_n \) is the set of all points on a sphere of radius \( \sqrt{\kappa} \) in
\( \mathbb{R}^{n+1} \). Then the metric is obtained from the metric \( \text{d}_S \) for \( \mathbb{S}^n \) by
\( \text{d}_{\kappa>0}(x, y) = \text{d}_S\left(\frac{x}{\sqrt{\kappa}}, \frac{y}{\sqrt{\kappa}}\right) \). So the metric still corresponds to the angle between the two
points on the sphere. By the same argument, for \( \kappa < 0 \), \( M^\kappa_n \) is the set of
points \( \{x \in \mathbb{E}^{n,1} \mid <x|x> = \kappa \text{ and } x_{n+1} > 0\} \) with metric
\( \text{d}_{\kappa<0}(x, y) = \text{d}_H\left(\frac{x}{\sqrt{-\kappa}}, \frac{y}{\sqrt{-\kappa}}\right) \), where \( \text{d}_H \) is the metric defined for hyperbolic space.

Thus we have

Definition 1.6.1 For \( \kappa \in \mathbb{R} \), define the Model Space \( M^\kappa_n \) as follows:

1. for \( \kappa = 0 \), \( M^0_n = \mathbb{E}^n \)
2. for $\kappa > 0$, $M^n_\kappa$ is obtained from $\mathbb{S}^n$ by scaling the metric by $\frac{1}{\sqrt{\kappa}}$.

3. for $\kappa < 0$, $M^n_\kappa$ is obtained from $\mathbb{H}^n$ by scaling the metric by $\frac{1}{\sqrt{-\kappa}}$.

We might also know these as the Reimannian manifolds of constant curvature $\kappa$, but we prefer to think of them as metric spaces. We then have that for $\kappa \leq 0$, $M^n_\kappa$ is uniquely geodesic, and $\frac{\pi}{\sqrt{\kappa}}$-uniquely geodesic for $\kappa > 0$. Angles are still the Alexandrov angles. By rescaling, we get the “cosine rules” for $M^n_\kappa$ for any $\kappa \in \mathbb{R}$:

**Proposition 1.6.1 (The Cosine Rule for $M^n_\kappa$)**

Consider the geodesic triangle in Figure 1.1, in model space. Then

1. for $\kappa = 0$, $c^2 = a^2 + b^2 - 2ab\cos \gamma$
2. for $\kappa > 0$, $\cos(\sqrt{\kappa}c) = \cos(\sqrt{\kappa}a) \cos(\sqrt{\kappa}b) + \sin(\sqrt{\kappa}a) \sin(\sqrt{\kappa}b) \cos \gamma$
3. for $\kappa < 0$, $\cosh(\sqrt{-\kappa}c) = \cosh(\sqrt{-\kappa}a) \cosh(\sqrt{-\kappa}b) - \sinh(\sqrt{-\kappa}a) \sinh(\sqrt{-\kappa}b) \cos \gamma$

Note that if you fix $a, b$ and $\kappa$, then $c$ is a strictly increasing function of $\gamma$. This observation will prove to be useful as we proceed. The formulae 2 and 3 are really the same thing; by elementary complex number theory we can interchange them.

We use the cosine rule in $M^n_\kappa$ to prove the following:

**Proposition 1.6.2 (Alexandrov’s Lemma)**

Take a geodesic quadrilateral $(A, B, C, B')$ in $M^n_\kappa$, perimeter $< \frac{2\pi}{\sqrt{\kappa}}$ if $\kappa > 0$. Suppose $B$ and $B'$ lie on opposite sides of the geodesic segment $[A, C]$.

Consider the geodesic triangles $\Delta = \Delta(A, B, C)$ and $\Delta' = \Delta(A, B', C)$. Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be the interior angles at $A, B, C, A', B', C'$ respectively. Assume that $\gamma + \gamma' \geq \pi$. 

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Then

1. $d(B, C) + d(B', C) \leq d(B, A) + d(B', A)$.

Let $\triangle$ be a triangle in $M^n_\kappa$ with vertices $\overline{A}, \overline{B}, \overline{B}'$ such that $d(\overline{A}, \overline{B}) = d(A, B)$, $d(\overline{A}, \overline{B}') = d(A, B')$, $d(\overline{B}, \overline{B}') = d(B, C) + d(C, B')$. That is, a “comparison triangle” for the vertices $A, B, B'$ of the quadrilateral. Note that we require $d(B, C) + d(C, B') < \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ so that $[\overline{B}, \overline{B}']$ exists. Let $C$ be the point of $[\overline{B}, \overline{B}']$ such that $d(B, C) = d(B, C)$. Let $\alpha, \beta, \beta'$ be the angles of $\triangle$ at the vertices $\overline{A}, \overline{B}, \overline{B}'$.

Then

2. $\bar{\alpha} \geq \alpha + \alpha'$, $\bar{\beta} \geq \beta$, $\bar{\beta}' \geq \beta'$ and $d(A, C) \leq d(\overline{A}, \overline{C})$, with equality if and only if $\gamma + \gamma' = \pi$.

Proof: Let $\bar{B}' \in M^n_\kappa$ be the unique point such that $d(\bar{B}', C) = d(B, C)$ and the geodesic $[B, \bar{B}']$ contains $[B, C]$. Since $\gamma + \gamma' \geq \pi$, the angle $\angle_{\overline{C}}(A, \bar{B}')$ is no greater than $\angle_{\overline{C}}(A, B')$. 

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Then by the cosine rule, \( d(\tilde{B}', A) \leq d(B', A) \). Therefore \( d(B, A) + d(B', A) \geq d(B, A) + d(\tilde{B}', A) = d(B, \tilde{B}') = d(B, C) + d(C, \tilde{B}') = d(B, C) + d(C, B') \). This proves 1.

To obtain the “comparison triangle”, we imagine that the vertices \( A, B, C, B' \) have hinges, and we straighten the bottom edge to get a vertex angle of \( \pi \) at \( C \). More formally, from 1 we have \( d(\tilde{B}', A) \leq d(B', A) = d(B', A) \). By the triangle inequality, we also have \( d(B, \tilde{B}') = d(B, C) + d(C, B') \). In each case the inequality is strict unless \( C \in [B, B'] \), when \( \gamma + \gamma' = \pi \).

Then applying the cosine rule we get \( \alpha \geq \alpha' + \alpha' \), \( \beta \geq \beta' \). Exchanging the roles of \( \beta \) and \( \beta' \) we get \( \beta' \geq \beta' \).

Then by the cosine rule again \( d(A, C) \leq d(A, C) \). It is clear that we have equality only when \( \gamma + \gamma' = \pi \), in which case the quadrilateral is already a triangle, so no straightening need occur.

We want to consider the model spaces because they have nice concrete properties like angles and curvature. We are going to “compare” an arbitrary metric space to model spaces and exploit these properties, to get information back about our space.

1.7 The CAT(\( \kappa \)) Inequality

We needn’t restrict our comparison of triangles in a geodesic metric space to those in \( \mathbb{E}^2 \). In fact it is fruitful to consider comparison triangles in any of our model spaces \( M_\kappa^n \); if triangles in \( X \) look like those in \( M_\kappa^n \) for some \( \kappa \), then we can relate some kind of “curvature” back to \( X \). This relation will be the CAT(\( \kappa \)) inequality.
Define a comparison triangle in $M^n_\kappa$ analogously with one in $\mathbb{E}^2$. For a geodesic triangle $\triangle([p, q], [q, r], [r, p])$ in $X$, there are comparison points $\overline{p}, \overline{q}$ and $\overline{r}$ in $M^n_\kappa$ such that $d(p, q) = d(\overline{p}, \overline{q})$, $d(q, r) = d(\overline{q}, \overline{r})$, and $d(r, p) = d(\overline{r}, \overline{p})$. These points are unique in $M^n_\kappa$ up to isometry, and have unique geodesics between them, provided the sum of their lengths is less than $\frac{2\pi}{\sqrt{\kappa}}$ when $\kappa > 0$. That is, consider the sphere of radius $1/\sqrt{\kappa}$:

![Figure 1.10](image)

A triangle of perimeter greater than the circumference of this sphere will not fit.

We denote the comparison triangle by $\overline{\triangle}([p, q], [q, r], [r, p])$, but since geodesics in $M^n_\kappa$ are unique (modulo restrictions for $\kappa > 0$), we can write this as $\triangle(\overline{p}, \overline{q}, \overline{r})$. If $x \in [q, r]$ then $\overline{x} \in [\overline{q}, \overline{r}]$ such that $d(q, x) = d(\overline{q}, \overline{x})$ is the comparison point for $x$. We retain our concept of angle as the Alexandrov angle in our spaces, so in each case we compare to a triangle in $\mathbb{E}^2$.

**Definition 1.7.1** Let $\kappa \in \mathbb{R}$.

We say a geodesic triangle $\triangle([p, q], [q, r], [r, p])$ satisfies the CAT($\kappa$)-inequality if, given its comparison triangle $\overline{\triangle}([p, q], [q, r], [r, p])$ in $M^n_\kappa$, the following inequality is satisfied for any point $x \in [q, r]$:

$$d(p, x) \leq d(\overline{p}, \overline{x})$$
That is:

![Figure 1.11:](image)

We say $X$ is a CAT($\kappa$) space, or simply CAT($\kappa$), if every geodesic triangle in $X$ (of perimeter $\leq \frac{2\pi}{\sqrt{\kappa}}$ when $\kappa > 0$) satisfies the CAT($\kappa$)-inequality.

Intuitively, this condition says that triangles are no “fatter” than those in the comparison model space $M^n_{\kappa}$. We say that $X$ has “curvature” no greater than that of $M^n_{\kappa}$, ie. $\kappa$, if $X$ is a CAT($\kappa$) space locally. This serves as a definition for curvature in an arbitrary geodesic metric space.

The acronym CAT stands for Cartan, Alexandrov and Toponogov, each of whom having made a substantial contribution to the theory.

We can characterise CAT($\kappa$) spaces in a number of equivalent ways:

**Proposition 1.7.1** Let $X$ be a geodesic metric space, $\kappa \in \mathbb{R}$. The following are equivalent:

1. $X$ is CAT($\kappa$)

2. For every geodesic triangle $\triangle([p, q], [q, r], [r, p])$ in $X$ and for each pair of points $x, y \in \triangle$ with comparison points $\overline{x}, \overline{y} \in \overline{\triangle} \subseteq M^2_{\kappa}$, the following inequality is satisfied:
   
   $$d(x, y) \leq d(\overline{x}, \overline{y}).$$

3. For every geodesic triangle $\triangle([p, q], [q, r], [r, p])$ in $X$ and for each pair of points $x \in [p, q], y \in [p, r], x \neq y$, the following inequality is satisfied by the angles $\overline{\alpha}, \overline{\alpha'}$ in the corresponding comparison triangles for $(p, q, r)$ and $(p, x, y)$ respectively in $M^2_{\kappa}$:

   $$\overline{\alpha'} \leq \overline{\alpha}$$

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4. The Alexandrov angle at a vertex of any geodesic triangle in $X$ is no greater than that of the corresponding comparison triangle in $M^2_\kappa$.

Discussion: Conditions 3 and 4 are saying that as you get closer to a vertex, your triangle doesn’t get “fatter”. It is clear that $3 \Rightarrow 4$, by definition of the Alexandrov angle in $X$.

It is also clear that $2 \Rightarrow 1$, by putting $y=q$. 2 appears more general, but each condition is in fact equivalent.

Proof: $2 \iff 3$ is a consequence of the cosine rule in $M^2_\kappa$. Let $p, q, r, \alpha, \alpha'$ be as in 3. Consider the comparison triangles $\overline{\triangle} = \triangle(p, q, r)$ and $\overline{\triangle}' = \triangle(p', x', y')$ as in Figure 1.12 in $M^2_\kappa$, where $d(q, r) = d(q, r)$, $d(x, y) = d(x', y')$, etc. If $x$ and $y$ lie on the same geodesic segment, then we get equality, ie. $d(x, y) = d(x, y)$ from $\overline{\triangle}$. If not, choose $p$ to be the vertex between the segments containing $x$ and $y$. Now $d(p, x) = d(p, x) = d(p', x')$, and same for $y$, so these side lengths and $\kappa$ are constant. Thus by the cosine rule (1.6.1), the angle at $p$ is a strictly increasing function of the length of the opposite side, hence

$$d(x, y) \geq d(x, y) = d(x', y')$$

if and only if $\alpha \geq \alpha'$.

$1 \Rightarrow 3$: Let $p, q, r, \alpha, \alpha'$ be as in 3. Consider the comparison triangle $\triangle(p'', x'', y'')$ for $\triangle([p, x], [x, r], [r, p])$ in $M^2_\kappa$, where $[p, x] \subset [p, q]$. Let $\alpha''$ be the angle at $p''$. 

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By 1, we have $d(x', y') = d(x, y) \leq d(x'', y'')$ where $y'' \in [p'', r'']$ is the comparison point for $y$. Then, by the cosine rule, $\alpha' \leq \alpha''$. By applying 1 again, $d(x'', r'') = d(x, r) \leq d(x, r)$ comparing $\triangle(p'', x'', r'')$ to $\triangle(p, x, r)$. Thus $\alpha'' \leq \alpha$. Putting these together, $\alpha' \leq \alpha'' \leq \alpha$.

$4 \Rightarrow 1$: Let $\triangle([p, q], [q, r], [r, p])$ be a triangle in $X$, and $x \in [q, r]$. Fix a geodesic $[p, x]$ from $p$ to $x$. Let $\gamma, \gamma'$ be the interior angles at $x$, and $\beta$ the angle at $q$. Consider the comparison triangle $\overline{\triangle}(p, q, r)$ with comparison angle $\overline{\beta}$ at the comparison point $\overline{q}$ for $q$.

Let $\overline{\Delta}_1 = \triangle(p', q', x'), \overline{\Delta}_2 = \triangle(p', x', r')$ be comparison triangles for the triangles $\triangle([p, q], [q, x], [x, p])$, $\triangle([p, r], [r, x], [x, p])$ respectively. We can assume that $q', r'$ lie on opposite sides of $[p', x']$ in $M^2_{\kappa}$. Take comparison angles $\overline{\beta}, \overline{\gamma}$ in $\overline{\Delta}_1$ and $\overline{\gamma}'$ in $\overline{\Delta}_2$.

From Proposition 1.3.1 we have $\gamma + \gamma' \geq \pi$, since $x$ is on a geodesic. So by 4, $\overline{\gamma} + \overline{\gamma}' \geq \gamma + \gamma' \geq \pi$. 

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Then by Alexandrov’s lemma (1.6.2), straightening the quadrilateral \( \overline{\triangle_1 \cup \triangle_2} \) to \( \overline{\Delta} \) means \( \beta' \leq \beta \).

Then by the cosine rule in \( M^2_\kappa \) (1.6.1), \( d(p, x) \geq d(p', x') = d(p, x) \) \( \Box \)

**Proposition 1.7.2** Let \( X \) be a CAT(\( \kappa \)) space. Then \( X \) is uniquely geodesic for \( \kappa \leq 0 \), and \( \frac{\pi}{\sqrt{\kappa}} \)-uniquely geodesic for \( \kappa > 0 \).

Proof: Suppose that for some \( p, q \in X \), with \( d(p, q) \leq \frac{\pi}{\sqrt{\kappa}} \) if \( \kappa > 0 \), \( [p, q], [p, q]' \) are two geodesics from \( p \) to \( q \). Then form the bigon

\[
\begin{array}{c}
p \\
\text{[pq]} \quad \text{[pq]}
\end{array}
\]

\[
\begin{array}{c}
p \\
\text{[pq]}
\end{array}
\]

**Figure 1.16:**

We can consider this as a geodesic triangle \( \triangle([p, p], [p, q], [p, q]') \) which has a degenerate comparison triangle since \( [p, q] \) is unique in \( M^2_\kappa \).

\[
\begin{array}{c}
p \\
\text{[pq]}
\end{array}
\]

\[
\begin{array}{c}
p \\
\text{[pq]}
\end{array}
\]

**Figure 1.17:**

Thus if \( x \in [p, q], x' \in [p, q]' \) such that \( d(p, x) = d(p, x') \), then \( d(x, x') \leq d(x, x') = 0 \) by 2 above.

Hence the triangle (bigon) in \( X \) is also degenerate, so the geodesic from \( p \) to \( q \) is unique. \( \Box \)

**Proposition 1.7.3** Let \( X \) be a geodesic metric space, and \( \kappa \leq \kappa' \) real numbers. Then \( X \) is CAT(\( \kappa \)) implies it is CAT(\( \kappa' \)).

Proof: If \( \kappa > 0 \), then geodesic triangles satisfying the inequality are assumed to have perimeter \( < \frac{2\pi}{\sqrt{\kappa}} \), so they obviously have perimeter \( < \frac{2\pi}{\sqrt{\kappa'}} \) since \( \kappa \leq \kappa' \). Then we need only observe that for two comparison triangles \( \overline{\Delta} \) in \( M^\mu_\kappa \) and \( \overline{\Delta'} \) in \( M^\mu_{\kappa'} \), the vertex angles in \( \overline{\Delta} \) are less than the corresponding ones in \( \overline{\Delta'} \). Then by Proposition 1.7.1 4, we have our result. \( \Box \)

(See \([2]\) II p.7 for full proof.)

**Proposition 1.7.4** Let \( X \) be a CAT(\( \kappa \)) space.

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1. Let \( p, x, y \in X \) such that \( d(x, p) + d(p, y) \leq \frac{\pi}{\sqrt{\kappa}} \) if \( \kappa > 0 \). Then \( [x, y] = [x, p] \cup [p, y] \Leftrightarrow \angle p(x, y) = \pi. \)

2. A path \( c : [0, b] \to X \) of length \( \leq \frac{\pi}{\sqrt{\kappa}} \) if \( \kappa > 0 \) is a geodesic if and only if it is a local geodesic.

Proof: 1 \((\Rightarrow)\) By definition of angle, the comparison triangle in \( \mathbb{E}^2 \) for \( \triangle([x, y], [x, p], [p, y]) \) is degenerate so comparison angles are all \( \pi \). Thus the limsup of these is \( \pi \).

\((\Leftarrow)\) Since \( X \) is CAT(\( \kappa \)), the Alexandrov angle at \( p \) is no greater than the angle of the corresponding comparison triangle in \( M_\kappa^2 \).

So for the triangle above the angle \( \angle p(x, y) = \pi \). But triangles in \( M_\kappa^2 \) can’t have vertex angles greater than \( \pi \) (with the appropriate lengths if \( \kappa > 0 \)), so the angle at \( p \) in comparison triangles in \( M_\kappa^2 \) is always \( \pi \), so all comparison triangles are degenerate in \( M_\kappa^2 \). Thus by the CAT(\( \kappa \)) inequality, \( \triangle([x, y], [x, p], [p, y]) \) must be degenerate, hence the result.

2: Recall that \( c : [0, b] \to X \) is a local geodesic if \( \forall t \in (0, b) \exists \epsilon > 0 \) such that \( c|_{(t-\epsilon, t+\epsilon)} \) is a geodesic. Then clearly geodesic implies local geodesic.

Suppose then that \( c \) is a local geodesic. Let \( t_0 \) be the supremum of the \( t \in [0, b] \) so that \( c|_{[0, t]} \) is a geodesic.

Suppose \( t_0 \neq b \). Then \( \exists \epsilon_0 > 0 \) such that \( c|_{[t_0-\epsilon_0, t_0+\epsilon_0]} \) is a geodesic.

So we have geodesic segments \( [c(t_0-\epsilon_0), c(t_0+\epsilon_0)] = [c(t_0-\epsilon_0), c(t_0)] \cup [c(t_0), c(t_0+\epsilon_0)] \) so by 1 the angle at \( c(t_0) \) is \( \pi \).

Recall that the angle only depends on the germ of the geodesics at a point.

We know \( c|_{[0, t_0]} \) and \( [c(t_0), c(t_0+\epsilon_0)] \) are geodesics, and the angle at \( c(t_0) \) is \( \pi \).

Now since \( X \) is a CAT(\( \kappa \)) space, there is a unique geodesic from \( c(0) \) to \( c(t_0+\epsilon_0) \). Then by 1 again, this must be the union of \( [c(0), c(t_0)] \) and \( [c(t_0), c(t_0+\epsilon_0)] \), so \( c|_{[0, t_0+\epsilon_0]} \) is a geodesic, contradicting that \( t_0 \) is the supremum. \( \square \)

**Proposition 1.7.5 (Gluing Lemma)**

Let \( \triangle([p, q_1], [p, q_2], [q_1, q_2]) \) be a geodesic triangle in a geodesic metric space \( X \). Let \( r \in [q_1, q_2] \), and let \([p, r] \) be a geodesic joining \( p \) and \( r \). Take \( \kappa \in \mathbb{R}; \) if \( \kappa > 0 \), suppose the perimeter of the triangle is less than \( \frac{2\pi}{\sqrt{\kappa}} \).

Then if the triangles \( \triangle([p, q_1], [p, r], [q_1, r]) \) and \( \triangle([p, q_2], [p, r], [q_2, r]) \) are each CAT(\( \kappa \)), then so is \( \triangle([p, q_1], [p, q_2], [q_1, q_2]) \).
Proof: Let $\Delta_1, \Delta_2$ be comparison triangles for $\Delta_1 = \triangle([p, q_1], [p, r], [q_1, r])$, $\Delta_2 = \triangle([p, q_2], [p, r], [q_2, r])$ respectively, and assume $\overline{q_1}, \overline{q_2}$ are on opposite sides of the segment $[p, r]$. The interior angle at $r$ in $\triangle([p, q_1], [p, q_2], [q_1, q_2])$ is $\pi$ since it lies on a geodesic.

Let $\gamma_1, \gamma_2$ be the angles at $r$ in $\Delta_1, \Delta_2$, and $\overline{\gamma_1}, \overline{\gamma_2}$ be comparison angles in $\overline{\Delta_1}, \overline{\Delta_2}$.

Since each is CAT($\kappa$), $\pi = \gamma_1 + \gamma_2 \leq \overline{\gamma_1} + \overline{\gamma_2}$.

So we can apply Alexandrov's lemma (1.6.2) to "straighten" the two comparison triangles, to obtain a comparison triangle $\closure{\triangle}$ for $\triangle([p, q_1], [p, q_2], [q_1, q_2])$

Since the other angles in the quadrilateral increase as $\gamma_1 + \gamma_2$ decreases, then the comparison angles in $\closure{\triangle}$ will satisfy condition 4 of Proposition 1.7.1.

That is,

$$\alpha_i \leq \overline{\alpha_i}, \beta_i \leq \overline{\beta_i}$$ by the CAT($\kappa$) inequality for each triangle, and $\overline{\alpha_i'} \geq \overline{\alpha_i}$, $\overline{\beta'} = (\overline{\beta_1} + \overline{\beta_2})' \geq \overline{\beta_1} + \overline{\beta_2}$.

Hence $\alpha_i \leq \overline{\alpha_i}'$ and $\beta = \beta_1 + \beta_2 \leq \overline{\beta'}$.

□
Recall that a space $X$ has curvature $\leq \kappa$ if every point $x \in X$ has a neighbourhood $U_x$ that is CAT($\kappa$). Under further conditions, this local property becomes a global one.

**Proposition 1.7.6** Suppose a geodesic metric space $X$ is locally CAT($\kappa$), and $\forall p, q \in X$ with $d(p, q) < \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$, $[p, q]$ is unique in $X$. Then $X$ is CAT($\kappa$).

Proof: See Ballman [1], p.193.

We know from Proposition 1.7.2 that CAT($\kappa$) spaces are uniquely geodesic or $\frac{\pi}{\sqrt{\kappa}}$-uniquely geodesic for $\kappa > 0$. It is clear that CAT($\kappa$) implies CAT($\kappa$) locally, therefore the above statement is in fact if and only if.

The following theorem is a generalised version of the Cartan-Hadamard Theorem in Riemannian geometry.

**Theorem 1.7.7** (Cartan-Hadamard)

Let $X$ be a geodesic metric space which is locally CAT($\kappa$). Let $p, q \in X$. Then each homotopy class of curves from $p$ to $q$ contains exactly one geodesic.

Proof: See Ballman [1], p.195.

**Corollary 1.7.8** If $X$ is simply connected and locally CAT($\kappa$), then $X$ is CAT($\kappa$).

Proof: If $X$ is simply connected, then there is only one such homotopy class of curves between any two points in $X$. Thus $X$ is uniquely geodesic. So combined with Proposition 1.7.6 we have that $X$ is CAT($\kappa$).

Another good geometric proof of this important result can be found in Paulin [11]. The main idea is to subdivide a geodesic triangle into sufficiently small ones. Then by certain facts about unique geodesics varying with their endpoints, we can reglue them to obtain a CAT($\kappa$) triangle.

### 1.8 Cones Over Metric Spaces

**Definition 1.8.1** Let $Y$ be a metric space, $\kappa \in \mathbb{R}$. The $\kappa$-cone over $Y$, $C_\kappa Y$, is the space given by $[0, \infty) \times Y / \sim$ for $\kappa \leq 0$, $[0, \frac{\pi}{2\sqrt{\kappa}}] \times Y / \sim$ for $\kappa > 0$, where $(t, y) \sim (t', y')$ if $t = t' = 0$ or $t = t' > 0$ and $y = y'$.
Geometrically, $C_\kappa Y$ can be thought of as:

![Figure 1.20](image1.png)

We denote $(t,y)$ as $ty$ if $t > 0$, and 0 if $t = 0$. $(0,y) = 0$ is called the cone point of $C_\kappa Y$. To define a metric on $C_\kappa Y$, thus making it a metric space itself, we first declare the angle at the cone point between the rays to two points $y_1, y_2$ in $Y$ to be the distance between them in $Y$.

![Figure 1.21](image2.png)

If $d(y_1, y_2) \geq \pi$ in $Y$, this angle won’t be much use in terms of our geometric model, so define $d_\pi(y_1, y_2) = \min\{\pi, d(y_1, y_2)\}$. Let $x_1 = t_1y_1$, $x_2 = t_2y_2$ be two points in $C_\kappa Y$. Then $x_i$ lies along the ray from the cone point to $y_i$ in $Y$, at distance $t_i$ from the cone point.

![Figure 1.22](image3.png)

Then by copying the cosine rules in $M^n_\kappa$, we can define a metric on $C_\kappa Y$ as follows:
For $\kappa = 0$, 
\[ d(x_1, x_2)^2 = t_1^2 + t_2^2 - 2t_1t_2\cos(d_\pi(y_1, y_2)) \]

For $\kappa > 0$, 
\[ \cos(\sqrt{\kappa}d(x_1, x_2)) = \cos(\sqrt{\kappa}t_1) \cos(\sqrt{\kappa}t_2) + \sin(\sqrt{\kappa}t_1) \sin(\sqrt{\kappa}t_2) \cos(d_\pi(y_1, y_2)) \]

For $\kappa < 0$, 
\[ \cosh(\sqrt{-\kappa}d(x_1, x_2)) = \cosh(\sqrt{-\kappa}t_1) \cosh(\sqrt{-\kappa}t_2) - \sinh(\sqrt{-\kappa}t_1) \sinh(\sqrt{-\kappa}t_2) \cos(d_\pi(y_1, y_2)) \]

From the geometric representation of a cone, this seems a natural way to induce distances on $C_\kappa Y$ that will obey the conditions for a metric.

We note immediately that for $t_1, t_2 > 0$, $d(x_1, x_2) = t_1 + t_2 \Leftrightarrow d_\pi(y_1, y_2) \geq \pi$, i.e., $d_\pi = \pi$. That $d(x_1, x_2)$ is a metric is then easy except for the triangle inequality. (See [2] I p.45)

**Proposition 1.8.1** Let $x_1 = t_1y_1, x_2 = t_2y_2 \in C_\kappa Y$.

1. If $t_1, t_2 > 0$ and $d(y_1, y_2) < \pi$, then there is a bijection between the set of geodesic segments $[y_1, y_2]$ in $Y$, and the set of geodesic segments $[x_1, x_2]$ in $C_\kappa Y$.

2. Otherwise, $\exists! [x_1, x_2]$ in $C_\kappa Y$ (except possibly when $\kappa > 0$ and $d(x_1, x_2) = \frac{\pi}{\sqrt{\kappa}}$).

Proof: 1: We say $C_\kappa Y'$ is a *subcone* of $C_\kappa Y$ if $Y'$ is a subset of $Y$. For a geodesic segment $[y_1, y_2] \subseteq Y$, the subcone $C_\kappa [y_1, y_2] \subseteq C_\kappa Y$ is isometric to a convex sector in $M^2_\kappa$. Thus there is a geodesic joining $x_1$ to $x_2$.

Conversely, take a geodesic segment $[x_1, x_2] \subseteq C_\kappa Y$ and let $x = ty$ be any point along it. If $t = 0$ then we would have $d(x_1, x_2) = t_1 + t_2 \Rightarrow d(y_1, y_2) \geq \pi$.

So $t \neq 0$, so the projection $x = ty \mapsto y$ of $[x_1, x_2]$ into $Y$ is well defined.

So we need to prove the image of $[x_1, x_2]$ under this projection is geodesic in $Y$. It suffices to show $d(y_1, y) + d(y, y_2) = d(y_1, y_2)$ for arbitrary $y$.

Consider the comparison triangle $\Delta_1 = \triangle(\bar{0}, \bar{x}_1, \bar{x})$ and $\Delta_2 = \triangle(\bar{0}, \bar{x}, \bar{x}_2)$, arranged so that $\bar{x}_1$ and $\bar{x}_2$ are on opposite sides of the common segment.

---

**Figure 1.23:**
The angles at the cone point correspond to lengths in $Y$. They are the same in the comparison triangles.

So we have $\theta_1 = d(y_1, y), \theta_2 = d(y, y_2)$. Now since $x \in [x_1, x_2]$ geodesic, $d(x_1, x) + d(x, x_2) = d(x_1, x_2) < t_1 + t_2 = d(0, x_1) + d(0, x_2)$.

This forces $\theta_1 + \theta_2 < \pi$, since the “top” two edges are longer than the “bottom” two.

Let $\Delta' = \Delta(0', x_1', x_2')$ be a comparison triangle in $M_2^2$ for $(0, x_1, x_2)$.

![Figure 1.24:](image)

The angle $\theta'$ at $0'$ is $d(y_1, y_2)$. By the cosine rule (1.1), since

$$d(x_1, x_2) \leq d(x_1, x) + d(x, x_2) = d(x_1, x_2) = d(x_1', x_2'),$$

then $\theta_1 + \theta_2 \leq \theta$.

![Figure 1.25:](image)

That is, $d(y_1, y) + d(y, y_2) \leq d(y_1, y_2)$. The triangle inequality gives us the reverse, hence our result.

2: If $d(y_1, y_2) \geq \pi$ or $t_i = 0$ then $d(x_1, x_2) = t_1 + t_2$. So the geodesic in $C_\kappa Y$ from $x_1$ to $x_2$ is $c : [0, t_1 + t_2] \to C_\kappa Y$ mapping $t \in [0, t_i]$ onto $(t_1 - t)y_1$ and $t \in [t_i, t_1 + t_2]$ onto $(t_2 - t)y_2$. We need to show that is is the only geodesic possible, provided $t_1 + t_2 < \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$.

If one of the $t_i$ is zero then this is clear. So assume $t_1, t_2 > 0$.

Suppose there is $x = ty$ such that $d(x_1, x) + d(x, x_2) = d(x_1, x_2)$ and $d(x_1, x) < t_1$. Then we must show $y = y_1$.

Consider the triangles $\Delta_1, \Delta_2$ as in 1.

Then $\pi \leq d(y_1, y_2) \leq d(y_1, y) + d(y, y_2)$ by the triangle inequality, so the angle at the cone point is obtuse or $\pi$. 

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But \( d(\vec{x}_1, \vec{0}) + d(\vec{0}, \vec{x}_2) = t_1 + t_2 = d(x_1x_2) = d(\vec{x}_1, \vec{x}) + d(\vec{x}, \vec{x}_2). \)

So both triangles must be degenerate, thus \( x = ty_1. \)

This proposition says that if geodesics are unique in one space, then they must be in the other. Further, we have the following theorem.

**Theorem 1.8.2 (Berestovskii)**

\( C_\kappa Y \) is CAT(\( \kappa \)) if and only if \( Y \) is CAT(1).

That is, \( Y \) needs to be spherical for the geometry to work. For example, an ideal case would be:

![Figure 1.28: Ideal case of CAT(1) space](image)

**Proof:** We know that CAT(\( \kappa \)) spaces have unique geodesics between points of certain distances apart. Then the above proposition implies that if \( C_\kappa Y \) is CAT(\( \kappa \)) then \( Y \) is \( \pi \)-uniquely geodesic and conversely if \( Y \) is CAT(1) then \( C_\kappa Y \) is uniquely geodesic (or \( \pi \sqrt{\kappa} \)-uniquely geodesic if \( \kappa > 0 \)). Then for any geodesic triangle in one space there corresponds a unique geodesic
triangle in the other. By constructing appropriate comparison triangles we can obtain our result.

Chapter 2

Metric Complexes and the Link Condition

2.1 Metric Complexes

Define an $\mathcal{M}_n^\kappa$-cell to be the metric space $(X, d_X)$ which is isometric to a compact subset of $\mathcal{M}_n^\kappa$ which is the intersection of finitely many closed half-spaces of $\mathcal{M}_n^\kappa$. Equivalently, we can define it as the convex hull of a set of points in $\mathcal{M}_n^\kappa$. A face of $X$ is either $X$ itself or $X \cap H$, where $H$ is a hyperplane disjoint from the interior of $X$. Thus faces are again $\mathcal{M}_m^\kappa$-cells for some $m \leq n$.

An $\mathcal{M}_n^\kappa$-complex is the pair $K = (U, X)$ consisting of a set $U$ and a collection $X$ of metric spaces $(X, d_X)$ where $X$ is a subset of $U$, called the cells of $K$, such that:

1. $U = \bigcup_{X \in X} X$

2. each $X \in X$ is an $\mathcal{M}_m^\kappa$-cell for values of $m \leq n$

3. if $Y$ is a face of $X$, then $y \in X$, and $d_Y = d_X|_{Y \times Y}$

4. if $X,Y$ are cells, then $X \cap Y$ is a face of either cell.

$K$ has a natural topology, with $N$ open if and only if $N \cap X$ is open for each $X \in X$.

3 says that metrics agree on faces.
For our purposes, we need only consider complexes for \( \kappa = 0 \) or 1, which we can call Euclidean or Spherical Complexes respectively. However we will continue to consider results for any \( \kappa \in \mathbb{R} \) throughout this chapter.

Further, when the cells are all the same kind of polyhedra, for instance simplices or \( n \)-cubes, we refer to \( K \) as a simplicial or cubical complex.

### 2.2 Simplicial Complexes

An abstract simplicial complex is a set \( V \) of vertices, together with a set \( S \) of nonempty finite subsets of \( V \), such that

1. \( \{v\} \in S \) for all \( v \in V \)
2. if \( B = \{v_1, \ldots, v_{n+1}\} \in S \) then every nonempty subset \( C \) of \( B \) is in \( S \).

A subset \( B \) is called an \( n \)-simplex if it has \( n + 1 \) vertices, and \( C \) is called a face of \( B \).

Let \( \kappa \in \mathbb{R} \). A geodesic \( n \)-simplex in \( M^n_\kappa \) is the convex hull of \( n + 1 \) points in general position in \( M^n_\kappa \); if \( \kappa > 0 \), these points lie in an open ball of radius \( \frac{\pi}{2 \sqrt{\kappa}} \).

An \( M^n_\kappa \)-simplicial complex is an abstract simplicial complex \( K \), such that each simplex is isometric to some geodesic \( n \)-simplex, and faces glued by isometries.

Each simplex is called a cell of \( K \). Each cell \( B \) has a metric \( d_B \) induced from \( M^n_\kappa \).

An \( m \)-chain between two points \( x, y \in K \) is an \( (m + 1) \)-tuple of points \( (x_0, \ldots, x_m) \) in \( K \), so that \( x = x_0, y = x_m \), and each segment \([x_i, x_{i+1}]\) is a geodesic segment inside a cell \( B(i) \) of \( K \). The length of this chain is \( \sum_{i=0}^{m-1} d_{B(i)}(x_i, x_{i+1}) \).

A chain is then a piecewise linear path in \( K \), so to define a metric \( d \) on \( K \), we take the infimum of chains from \( x \) to \( y \) in \( K \), or if there is no such chain, make \( d(x, y) = \infty \). This happens if and only if \( K \) is not connected.

Define the set \( \text{Shapes}(K) \) to be the set of different isometry types of simplices in \( K \). If \( \text{ Shapes}(K) \) is finite, then the above \( d \) is indeed a metric.

**Proof:** The function \( d \) can be defined on any piecewise \( M^n_\kappa \)-simplicial complex where each cell has a metric which agrees on faces. In this case \( d \) is a pseudometric and yields a geodesic metric space on the complex. We would like to have a complete geodesic metric space. The pseudometric is
seen to fail if there are not finitely many isometry types of cells, hence our extra condition. Details of this can be found in [2] I p.68.

**Definition 2.2.1** (Link)

Let $K$ be an abstract simplicial complex. Define the link of a vertex $v$ in $K$, $LK(v,K)$, as the abstract simplicial complex with vertices corresponding to edges $\{v,u_i\}$ in $K$, and $k$-simplices $\{u_1,\ldots,u_{k+1}\}$ if and only if $\{v,u_1,\ldots,u_{k+1}\}$ is a $(k+1)$ simplex of $K$.

Now consider $K$ as an $M^n_\kappa$-simplicial complex. Consider a small ball around a vertex $v$ of $K$. Since edges in $K$ must have positive length we can find some $\epsilon$ so that the ball $B(v,\epsilon)$ lies completely inside cells that contain $v$.

![Figure 2.1:](image)

Then the link $LK(v,K)$ of $v$ in $K$ as an abstract simplicial complex projects naturally onto the boundary of $B(v,\epsilon)$. We think of the points of this ball as lying in the tangent space of $v$. That is, the set of all directions (vectors) from $v$ into a simplex containing it. Now the tangent space of a point in model space is Euclidean, regardless of the curvature of the space, thus the ball of all vectors from $v$ of unit length say is exactly spherical. Thus $LK(v,K)$ has a natural spherical or $M_1$-structure. So $LK(v,K)$ is an $M_1$-simplicial complex.

If Shapes($K$) is finite, then Shapes($LK(v,K)$) must also be finite.

To define a metric on $LK(v,K)$, consider the $\kappa$-cone $C_\kappa(LK(v,K))$ on the link of a vertex in $K$. Then a ball of radius $\epsilon$ about its cone point is isometric to the ball of radius $\epsilon$ about $v$. (See Proposition 7.37 [2] I p.76).

Thus the distance between two points on $LK(v,K)$ should correspond to the angle at the cone point, that is, the angle at $v$ in $K$.

### 2.3 Cubical Complexes

Let $I_n = \{1,\ldots,n\}$.
Define the vertex set for an abstract $n$-cube to be $I_n^{(0,1)}$, that is, the set of maps from $\{1,\ldots,n\}$ to $\{0,1\}$.

The geometric idea is that a “corner” of an $n$-cube is given by co-ordinates:

![Figure 2.2](image)

A codimension $k$ face of a cube is a choice of subset $J \subseteq I_n$ and a choice of map $\alpha$ from $J$ to $\{0,1\}$, so that the face is the set $\{f \in I_n^{(0,1)} | f|_J = \alpha\}$.

In other words, a face is obtained by holding some set of positions constant at either 0 or 1, and varying over the remaining ones.

**Example.**

![Figure 2.3](image)

$I_3^{(0,1)} = \{f : \{1,2,3\} \rightarrow \{0,1\}\}$

The front face is given by choosing $J = \{2\} \subset \{1,2,3\}$ and $\alpha : \epsilon_2 \rightarrow 0$, so that the face is the set of maps $f : \{1,2,3\} \rightarrow (\epsilon_1,0,\epsilon_3)$ with $\epsilon_i \in \{0,1\}$.

Combinatorially, the link of a vertex $v$ in this case is the abstract simplicial complex $LK(v,K)$, with vertices $\{u\}$ if and only if $\{v,u\}$ is an edge of $K$, and $k$-simplices $\{u_0,\ldots,u_n\}$ if and only if $\{v,u_0,\ldots,u_k\}$ is contained in a $(k+1)$-cube of $K$.

Geometrically, the link projects to the boundary of a ball of radius $\epsilon$ for some sufficiently small $\epsilon$ around a vertex $v$ of $K$. Again this ball is isometric to the $\kappa$-cone over $LK(v,K)$.

For example, in three dimensions:
An abstract cubical complex is a collection of cubes defined in this way, with some faces identified.

We define the link of a vertex to be the union of the links of $v$ for all the cells in $K$ that contain $v$. This collection of cells is known as the star of $v$. Of course, since the cells are glued together in a certain way in the complex, then the link is glued by restrictions of the same maps.

By similar a argument as in the simplicial case we have the following result. An $\epsilon$-neighbourhood of the cone point of the $\kappa$-cone over $LK(v, K)$ is isometric to an $\epsilon$-neighbourhood of $v$ in $L$. So the distance between points in the link is again given by the angle at $v$ made by the lines to $v$ in $L$.

### 2.4 The Link Condition

Take an $M_n^\kappa$-polyhedral complex to mean a simplicial or cubical one. We say an $M_n^\kappa$-polyhedral complex $K$ satisfies the link condition if for each vertex $v \in K$, $LK(v, K)$ is CAT(1). This condition is telling us how the pieces, that is, the polyhedra of $K$ are glued together. We know that each polyhedron of $K$ is itself CAT($\kappa$) if $K$ is an $M_n^\kappa$-polyhedral complex. If the link of all the vertices are CAT(1), then they fit together in a “nice” way. We will see that another characterisation of CAT(1) is that there is no closed geodesic of length less than $2\pi$, so the condition says, there are no “short cuts” around any of the vertices of $K$ when you have glued the polyhedra together. We make this explicit by the following theorem.

**Theorem 2.4.1** An $M_n^\kappa$-polyhedral complex $K$ is locally CAT($\kappa$) if and only if $K$ satisfies the link condition.

Proof: Take any vertex $v \in K$. We know that for a sufficiently small $\epsilon$, the ball $B(v, \epsilon)$ is isometric to the $\kappa$-cone over the link of $v$, $C_\kappa(LK(v, K))$, which is CAT($\kappa$). Then by Berestovskii’s Theorem (1.8.2), $LK(v, K)$ is CAT(1) $\iff C_\kappa(LK(v, K))$ is CAT($\kappa$). Thus, to complete the proof, since we know
the interiors of the polyhedra of $K$ are CAT($\kappa$), and a small neighbourhood of each vertex is CAT($\kappa$) if and only if each $LK(v, K)$ is CAT(1), we need only convince ourselves that points on the faces of polyhedra have neighbourhoods that are CAT($\kappa$). This is not hard. Given a small ball $B(v, \epsilon)$ at each vertex is CAT($\kappa$), we can homotope a ball of radius less than $\frac{1}{2}\epsilon$ about any point on a face to one inside $B(v, \epsilon)$, still on the face, not containing $v$. See Figure 2.5 Thus we have our result.

Figure 2.5: Metrically, $B(x, \frac{\epsilon}{3})$ and $B(x', \frac{\epsilon}{3})$ are the same.

**Proposition 2.4.2** Let $L$ be an $M_1^n$-simplicial complex. Then $L$ is CAT(1) if and only if $L$ contains no closed geodesics of length less than $2\pi$.

Proof: Let $C$ be a closed geodesic curve of length less than $2\pi$. Then choose two distinct points $x, y$ on $C$ to make a geodesic triangle $[x, y] \cup [y, x] \cup [x, x]$. If $L$ is CAT(1), then a comparison triangle for this bigon in $M_2^1$ is degenerate, so by the CAT(1) inequality, $C$ can’t exist.

Conversely, consider a geodesic triangle of perimeter less than $2\pi$ in $L$. Then this triangle must be null homotopic, for if not, it could homotope down to a closed geodesic of length less than $2\pi$, which isn’t allowed. Thus the homotopy from the triangle to a point is isometric to a “filled in” triangle. If it is entirely contained in one simplex of $L$, then we are done, since $L$ is $M_1$. Otherwise, consider the decomposition of the triangle by faces of simplices in $L$. Each piece is a smaller geodesic triangle entirely contained in a simplex, so is CAT(1), and by Proposition 1.7.5 gluing these back together gives CAT(1) triangles. Hence our result.

2.5 Flag Complexes

Next we consider another characterisation of the link condition. We say an $M_\kappa^n$-simplicial complex is a flag complex if every set of vertices pairwise joined
by an edge spans a simplex. That is, if \( \{v_i, v_j\} \) is an edge \( \forall i, j = 1, \ldots, n \), then \( \{v_1, \ldots, v_n\} \) is an \( (n-1) \)-simplex. It is clear that if \( L \) is a flag complex, then so is \( LK(v, L) \) for each vertex \( v \) of \( L \).

![Diagram of a simplex](image)

Figure 2.6:

That is, since \( \{u_1, \ldots, u_{n+1}\} \) is a simplex of \( LK(v, L) \) \( \Leftrightarrow \) \( \{u_1, \ldots, u_n, v\} \) is a simplex of \( L \), and vertices that are pairwise joined in \( L \) are pairwise joined in \( LK(v, L) \).

Then we have the following theorem.

**Theorem 2.5.1 (Gromov)**

*Let \( L \) be a finite dimensional \( M_1 \)-simplicial complex. If \( L \) is CAT(1) and has edges each of length \( \frac{\pi}{2} \), then \( L \) is a flag complex. If \( L \) is a flag complex whose edges are of length \( \geq \frac{\pi}{2} \), then \( L \) is CAT(1).*

Proof: Suppose \( L \) is CAT(1). Suppose there exist three pairwise joined vertices \( v_0, v_1, v_2 \) that do not span a simplex in \( L \).

Then in the link \( LK(v_0, L) \), the vertices corresponding to \( \{v_0, v_1\} \) and \( \{v_0, v_2\} \) in \( L \) are not joined by an edge, since there is no 2-simplex \( \{v_0, v_1, v_2\} \) in \( L \). Thus the distance between them is at least \( \pi \). That is, there must be at least two edges between them. The angle at \( v_0 \) between \( v_1 \) and \( v_2 \) in \( L \) is equal to the distance between the corresponding vertices in \( LK(v_0, L) \) in the cone over \( LK(v_0, L) \), which is \( d_\pi \) so equals \( \pi \). Thus, by Proposition 1.7.4, for some neighbourhood of \( v_0 \) in \( L \), \([x, v_0] \cup [v_0, y] = [x, y]\), so there is a geodesic locally around \( v_0 \). This applies at each vertex, so the closed path \( C = [v_0, v_1] \cup [v_1, v_2] \cup [v_2, v_0] \) is a local geodesic.

Then if \( L \) has edge lengths all \( \frac{\pi}{2} \), three such vertices cannot exist, for if so, divide the closed local geodesic \( C \) which has length \( \frac{3\pi}{2} \) into two local geodesic segments both of length less than \( \pi \). Then by Proposition 1.7.4, they are geodesics, so \( C \) is a geodesic bigon of length less than \( 2\pi \), giving us a contradiction. If \( L \) were not a flag complex, one can find some number of
vertices in $L$ that do not span a simplex. If more than three, consider the
link of one of them, say $v$. Then this would contain $n - 1$ vertices corre-
sponding to the edges from the other vertices to $v$, but there would be no
simplex spanned by these. Hence inductively, since $L$ is finite dimensional,
one can find three vertices not spanning a simplex somewhere. Hence $L$ is a
flag complex.

Now suppose $L$ is a finite dimensional $M_1$-simplicial complex, whose edges
are of length $\geq \frac{\pi}{2}$, and $L$ is a flag complex. We need to show that $L$ is CAT(1).
Consider the link of a vertex $v$ of $L$. Say $u_1, u_2$ are vertices of $LK(v, L)$.
Then we have $\{v, u_1\}, \{v, u_2\}$ edges in $L$. They are joined by an edge $\{u_1, u_2\}$
in $LK(v, L)$ if and only if there is a simplex $\{v, u_1, u_2\}$ in $L$. Then since $L$ is
$M_1$, the simplex $\{v, u_1, u_2\}$ is isometric to a triangle on the 2-sphere.

If $v$ is at the north pole, $v_1$ and $v_2$ must lie in the southern hemisphere or
the equator. The edge in $L$ between $u_1$ and $u_2$ has length $\geq \frac{\pi}{2}$, forcing the
vertex angle at $v$ to be $\geq \frac{\pi}{2}$.

Figure 2.7:

Then the length of the edge $\{u_1, u_2\}$ in $LK(v, L)$ is this angle. So we
have $LK(v, L)$ is $M_1$-simplicial, a flag complex, and has edges of length $\geq \frac{\pi}{2}$.
So we can use an inductive argument, on the dimension of $L$.

If $L$ is zero dimensional, it is a collection of vertices, so is trivially CAT(1)
since you can’t form geodesic triangles in $L$. Assume the proposition is true
for all $L$ of dimensions less than $n$. Then for $L$ an $M_1$-simplicial flag complex
with edges of length $\geq \frac{\pi}{2}$, the links of its vertices are all CAT(1).

Then by Berestovskii’s Theorem 1.8.2, the 1-cone over $LK(v, L)$ is CAT(1)
for all vertices $v$ of $L$. Thus $L$ is locally CAT(1).

Recall that a simplex of an $M_1$ complex is isometric to the convex hull of
its vertices, which must lie in a ball of radius $\frac{\pi}{2}$.

Consider a ball of radius $\frac{\pi}{2}$ about $v$ in $L$. Since $L$ is $M_1$, simplices are
all isometric to simplices on spheres. Then since edges are all length $\geq \frac{\pi}{2}$,
the ball $B(v, \frac{\pi}{2})$ is completely contained in the star of $v$. That is, it doesn’t extend past the faces opposite $v$.

Then this ball is isometric to a ball of radius $\frac{\pi}{2}$ about the cone point in $C_1(LK(v, L))$, which we know to be CAT(1).

Now consider a closed geodesic $C$ in $L$ of length $< 2\pi$.

$C$ cannot lie in a single simplex of $L$, since they are $M_1$, so CAT(1). So $C$ must cross a face of some adjacent simplices. Then since this face is a $M_1$ simplex, $C$ must be within $\frac{\pi}{2}$ of one of its vertices. In fact, any point on $C$ is always within $\frac{\pi}{2}$ of a vertex of the simplex containing it.

So $C \cap B(v, \frac{\pi}{2}) \neq \emptyset$ for some $v$ of $L$.

Since $C$ is a geodesic path, then this component $C'$ is a geodesic in $B(v, \frac{\pi}{2})$ which is CAT(1). Say $p$ and $q$ are the points on the boundary where $C$ enters and exits $B(v, \frac{\pi}{2})$. The comparison space for this ball is one hemisphere of a sphere.

Then if $d(p, q) \neq \pi$, we can form a bigon in $B(v, \frac{\pi}{2})$ which has degenerate comparison bigon in the 2-sphere:

![Figure 2.8](image)

Thus $C'$ must have length exactly $\pi$. Then $C$ cannot pass through two disjoint such balls, since its length is $< 2\pi$. When $C$ leaves $B(v, \frac{\pi}{2})$, it is within $\frac{\pi}{2}$ of another vertex.

So $C$ is contained in a collection of balls, which is finite since $L$ has only finitely many vertices. These balls pairwise overlap, so the vertices are pairwise contained in the same simplex, so are pairwise joined by edges.
Hence they must span a simplex in $L$, which means that $C$ is null homotopic. Therefore, there are no closed geodesics of length $< 2\pi$ in $L$, so $L$ is CAT(1).  $\Box$
Chapter 3

Coxeter Groups and the Moussong Complex

3.1 Coxeter Groups

\[ M = (m_{ij})_{1 \leq i, j \leq n} \] is a Coxeter Matrix if \( m_{ii} = 1, m_{ij} = m_{ji} \in [2, 3, \ldots, \infty) \) if \( i \neq j \).

For any Coxeter matrix there is a group presentation \( W(M) = W = \langle S | R \rangle \) with \( S = \{s_1, \ldots, s_n\} \), \( R = \{(s_i s_j)^{m_{ij}} | m_{ij} \in M\} \). If \( m_{ij} = \infty \), then there is no relation between the generators \( s_i, s_j \).

\( W \) is called a Coxeter Group if it has such a presentation. \((W,S)\) is called a Coxeter System. The rank of the system is \( n \).

\((W,S)\) can be represented as a labelled diagram \( D \), known as a Dynkin Diagram.

For each \( s_i \) draw a vertex. Then between each pair of vertices \( s_i, s_j \)

1. if \( m_{ij} = 2 \), put no edge
2. if \( m_{ij} \geq 3 \) put an edge.

Edges are labelled \( m_{ij} \) if \( m_{ij} > 3 \).

Example.

The group \( W = \langle s_1, s_2, s_3 | (s_1 s_2)^2, (s_1 s_3)^3, (s_2 s_3)^5 \rangle \) has the Dynkin diagram:
Dynkin diagrams are a very economical way to represent Coxeter groups. For any group $W$ having some generating set $S$ we can construct a *Cayley Graph* $\Gamma$; a labelled graph on which $W$ acts as a group action.

The vertices of $\Gamma$ correspond to the elements of $W$.

For each $w \in W$ and each $s \in S$, put an edge labelled $s$ between vertices $w$ and $ws$. We refer to this edge as $(w, ws)$.

Clearly $\Gamma$ is a finite graph if and only if $W$ is a finite group.

$W$ acts on the vertices of $\Gamma$ by left multiplication; $u \in W$ takes vertices $w$ to $uw$ and edges $(w, ws)$ to $(uw, uws)$.

The Cayley graph is the starting point for geometric group theory, in that it converts group theory into geometry. One can show that $W$ acts co-compactly discretely by isometries on its Cayley graph [9].

To make $\Gamma$ a metric space, we declare all edges to have length 1, and define a (metric): $d_\Gamma(w, w') = \min\{\text{len}(u) : w = uw' \text{ in } G\}$.

Since we are only looking at Coxeter Systems which have a fixed generating set, we can simplify this to $d_\Gamma(w, w') = l_S(w^{-1}w')$, where $l_S(w^{-1}w')$ is the number of generators in the shortest word for $w^{-1}w'$.

**Examples.**

$|S| = 0$, $W$ is the trivial group.

$|S| = 1$, $W \cong \mathbb{Z}_2$

$|S| = 2$, $W$ is the *dihedral group* of order $2m$; ie. $W = \langle s, t | s^2, t^2, (st)^m \rangle$

In general, $\Gamma$ is a $2m$-gon.
For $m = \infty$,

\[ |S| \geq 3 \text{ is where the interesting cases start.} \]
\[ W \text{ is a triangle group with presentation } < r, s, t | (rs)^{m_{rt}}, (st)^{m_{st}}, (rt)^{m_{rs}} > \]

For the simplest case of $m_{rs} = m_{st} = m_{rt} = 2$, we have

![Diagram](Figure 3.3)

We can construct a 2-dimensional complex from the Cayley graph by adding polygons, with edges labelled by generators, for every relation in the presentation. These polygons are glued by identifying the labelled edges with corresponding ones in $\Gamma$.

For a Coxeter system, this amounts to adding $2m_{ij}$-gons for each pair of generators $s_i, s_j$, provided $m_{ij} \neq \infty$.

**Examples.**

For $D_3$ we glue a hexagon:

![Diagram](Figure 3.5)

For $W = < r, s, t | (rs)^2, (st)^2, (rt)^2 >$ we glue six squares:

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We call this construction the Cayley Complex for $W$, denoted by $C$.

$W$ also acts on $C$ in a natural way, just as it does for $\Gamma$.

Now consider a closed loop in the Cayley complex. We can deform it onto the 1-skeleton, which is the Cayley graph. By isometry we can give it a base point at 1 in $\Gamma$. That is, if the loop passes through a vertex $w$, then act on $C$ by $w^{-1}$. Then this loop represents a relation, so it is a fairly standard result that it bounds 2-cells in the Cayley complex, so is homotopically trivial. Thus $C$ is simply connected.

### 3.2 The Reflection Representation

$(W, S)$ is a Coxeter System of rank $n$ with matrix $M = (m_{ij})_{1 \leq i, j \leq n}$.

Let $V$ be a real vector space, with basis $B = \{e_i\}_{i \in S}$. That is, $V = e_1 \mathbb{R} \oplus \ldots \oplus e_n \mathbb{R}$.

Let $B = (b_{ij})_{1 \leq i, j \leq n}$ be the matrix obtained from $M$ such that $b_{ij} = -\cos \frac{\pi}{m_{ij}}$. Note that $b_{ii} = 1$, $b_{ij} = b_{ji}$.

This induces a symmetric bilinear form on $V$ defined by $B(e_i, e_j) = b_{ij}$.

Thus for $x = \alpha_1 e_1 + \ldots + \alpha_n e_n$, $y = \beta_1 e_1 + \ldots + \beta_n e_n$, $B(x, y) = \sum_{1 \leq i, j \leq n} \alpha_i \beta_j B(e_i, e_j) = \sum_{1 \leq i, j \leq n} \alpha_i \beta_j b_{ij}$.

Recall $\sigma$ is a linear transformation if $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(\alpha x) = \alpha \sigma(x)$. Define a linear transformation $\sigma_i : V \to V$ by

$$\sigma_i(x) = x - 2B(x, e_i)e_i \ \forall x \in V.$$

We want $\sigma_i$ to be a representation of the group $W$, acting by reflections of $V$.

Thus, we would like $\sigma_i$ to have order 2, and $\sigma_i \sigma_j$ to have order $m_{ij}$.

$$\sigma_i(\sigma_i(x)) = \sigma_i(x - 2B(x, e_i)e_i)$$
\[
\begin{aligned}
&= x - 2B(x, e_i) e_i - 2B((x - 2B(x, e_i) e_i), e_i) e_i \\
&= x - 2B(x, e_i) e_i - 2B(x, e_i) e_i + 4B(x, e_i) B(e_i, e_i) e_i = x
\end{aligned}
\]

To consider \(\sigma_i \sigma_j\), recall that the composition of two linear transformations is again linear, and a linear transformation is completely determined by its action on the basis elements. Let \(V'\) be the subspace of \(V\) spanned by the basis \(B' = \{e_i, e_j\}\).

Then
\[
\sigma_i(e_i) = e_i - 2B(e_i, e_i) e_i = e_i - 2e_i = -e_i
\]
\[
\sigma_i(e_j) = e_j - 2B(e_j, e_i) e_i = e_j + 2\cos\frac{\pi}{m_{ij}} e_i
\]
\[
\sigma_i \sigma_j(e_i) = \sigma_i(e_i + 2\cos\frac{\pi}{m_{ij}} e_j)
\]
\[
= \sigma_i(e_i) + 2\cos\frac{\pi}{m_{ij}} \sigma_i(e_j)
\]
\[
= -e_i + 2\cos\frac{\pi}{m_{ij}}(e_j + 2\cos\frac{\pi}{m_{ij}} e_i)
\]
\[
= -e_i + 2\cos\frac{\pi}{m_{ij}} e_j + 4\cos^2\frac{\pi}{m_{ij}} e_i
\]
\[
= (4\cos^2\frac{\pi}{m_{ij}} - 1)e_i + 2\cos\frac{\pi}{m_{ij}} e_j.
\]
\[
\sigma_i \sigma_j(e_j) = \sigma_i(-e_j) = -\sigma_i(e_j) = -e_j - 2\cos\frac{\pi}{m_{ij}} e_i.
\]

In matrix form,
\[
\begin{bmatrix} \sigma_i \sigma_j(e_i) \end{bmatrix}_{V'} = \begin{bmatrix}
4\cos^2\frac{\pi}{m_{ij}} - 1 \\
2\cos\frac{\pi}{m_{ij}}
\end{bmatrix}
\]
\[
\begin{bmatrix} \sigma_i \sigma_j(e_j) \end{bmatrix}_{V'} = \begin{bmatrix}
-2\cos\frac{\pi}{m_{ij}} \\
-1
\end{bmatrix}
\]

Then the matrix of \(\sigma_i \sigma_j\) with respect to the bases \(B', B'\) is
\[
\begin{bmatrix} \sigma_i \sigma_j \end{bmatrix}_{B', B'} = \begin{bmatrix}
4\cos^2\frac{\pi}{m_{ij}} - 1 & -2\cos\frac{\pi}{m_{ij}} \\
2\cos\frac{\pi}{m_{ij}} & -1
\end{bmatrix}
\]

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\[ m_{ij} = \infty, \]
\[
\begin{bmatrix}
\sigma_i \sigma_j | V', B' &=& \begin{bmatrix}
3 & -2 \\
2 & -1
\end{bmatrix}
\end{bmatrix}
\]

Thus by induction
\[
([\sigma_i \sigma_j | V', B', B'])^n = \begin{bmatrix}
2n+1 & -2n \\
2n & -2n+1
\end{bmatrix} \neq \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \forall n > 0
\]

Thus \( \sigma_i \sigma_j \) has infinite order as required.

If \( m_{ij} < \infty \), we find the characteristic polynomial of \( M = [\sigma_i \sigma_j | V', B', B'] \):
\[
\det(\lambda I - M) = \begin{vmatrix}
\lambda - 4 \cos^2 \frac{\pi}{m_{ij}} + 1 & 2 \cos \frac{\pi}{m_{ij}} \\
-2 \cos \frac{\pi}{m_{ij}} & \lambda + 1
\end{vmatrix}
\]
\[
= (1 + \lambda)((1 + \lambda) - 4 \cos^2 \frac{\pi}{m_{ij}}) + 4 \cos^2 \frac{\pi}{m_{ij}}
\]
\[
= (1 + \lambda)^2 - 4 \cos^2 \frac{\pi}{m_{ij}}(1 + \lambda) + 4 \cos^2 \frac{\pi}{m_{ij}}
\]
\[
= (1 + \lambda)^2 - 4 \cos^2 \frac{\pi}{m_{ij}} \lambda = \lambda^2 + (2 - 4 \cos^2 \frac{\pi}{m_{ij}})\lambda + 1.
\]

This is zero when
\[
\lambda = \frac{1}{2}(4 \cos^2 \frac{\pi}{m_{ij}} - 2 \pm \sqrt{(2 - 4 \cos^2 \frac{\pi}{m_{ij}})^2 - 4})
\]
\[
= 2 \cos^2 \frac{\pi}{m_{ij}} - 1 \pm \sqrt{(1 - 2 \cos^2 \frac{\pi}{m_{ij}})^2 - 1}
\]
\[
= \cos \frac{2\pi}{m_{ij}} \pm \sqrt{(- \cos \frac{2\pi}{m_{ij}})^2 - 1} \text{ by double angle formula}
\]
\[
= \cos \frac{2\pi}{m_{ij}} \pm i \sin \frac{2\pi}{m_{ij}}
\]

which are the \( m_{ij} \)th roots of unity. Thus multiplying the matrix \( m_{ij} \) times gives the identity.

Hence \( [\sigma_i \sigma_j | V', B', B'] \) has order \( m_{ij} \), as required.

Thus we have a homomorphism \( s_i \mapsto \sigma_i \) between \( W \) and a subgroup of \( GL(V) \).
When $m_{ij}<\infty$, we also have $B|_{V'}(-,-)$ positive definite. That is $B|_{V'}(x,x)=0 \iff x=0$.

Proof: Say $x|_{V'}=a_i e_i + a_j e_j$. Then $B|_{V'}(x,x)=a_i B(e_i,x) + a_j B(e_j,x)$

$$= a_i(a_i B(e_i,e_i) + a_j B(e_i,e_j)) + a_j(a_i B(e_j,e_i) + a_j B(e_j,e_j))$$

$$= a_i^2 - 2a_i a_j \cos \frac{\pi}{m_{ij}} + a_j^2.$$ 

This is zero when $a_i=a_j (\cos \frac{\pi}{m_{ij}} \pm i \sin \frac{\pi}{m_{ij}})$ by similar calculation as above. But $a_i,a_j \in \mathbb{R}$ and $\sin \frac{\pi}{m_{ij}} \neq 0$ for $m_{ij}<\infty$, thus the only solution is $a_i=a_j=0$.

The hyperplane in $V \cong \mathbb{R}^n$ perpendicular to the subspace $V'$ is given by

$$(V')^\perp = \{ v \in V | B(v,v')=0 \ \forall \ v' \in V' \}.$$ 

Since $B$ is positive definite, clearly $V' \not\subseteq (V')^\perp$. $(e_i)^\perp = \{ v \in V | B(v,e_i)=0 \}$, so $(e_i)^\perp$ is codimension 1. Similarly for $e_j^\perp$, so $(V')^\perp$ is codimension 2.

It is clear that $(V')^\perp = (e_i)^\perp \cap (e_j)^\perp$.

Now $\sigma_i|_{(e_i)^\perp}$ is the identity, since $\sigma_i(x) = x - 2B(x,e_i)e_i = x \ \forall x \in (e_i)^\perp$. Similarly for $e_j^\perp$, so since $V = V' \oplus (V')^\perp$, and $\sigma_i \sigma_j$ has order $m_{ij}$ on $V'$ and is the identity on $(V')^\perp$, we have $\sigma \sigma_j$ has order $m_{ij}$ on $V$.

Then this representation induces an action of the group $W$ on $\mathbb{R}^n$, generated by reflections in the hyperplanes $(e_i)^\perp$ corresponding to generators $s_i$ for $W$.

Tits has proved that the representation is faithful [4]. That is, it is injective onto its image. Therefore, the map is an isomorphism. So we have made the group act directly as a group of reflections of real space. We will use this shortly to metrize the cells of the complex we wish to construct.

### 3.3 Special Subgroups

$(W,S)$ is a Coxeter system.

A subgroup of $W$ is called a special subgroup if it is generated by a subset of $S$. So for any $T \subseteq S$ we have the subgroup $< T >$. Of course we will have relations $R|_T$ for the generators in $T$, where $R|_T = \{(s_is_j)^{m_{ij}} | s_i,s_j \in T \}$. By a special result for Coxeter groups we have that these are the only relations, that is, the other relations in the larger group have no effect on the words
given in the generators of $T$. Thus we can denote the special subgroup by $W_T = \langle T \mid R \rangle_T$.

Then it is clear that $(W_T, T)$ is again a Coxeter system, with matrix $M_{|T \times T}$. We refer to $(W_T, T)$ as a Coxeter Subsystem of $(W, S)$.

Denote cosets of $W_T$ by $xW_T = \{xw \mid w \in W_T\}$.

Cosets of special subgroups have a nice intersection property:

**Lemma 3.3.1** Let $T, T' \subseteq S, x, y \in W$. Then $xW_T \cap yW_T'$ is either empty or of the form $zW_{T \cap T'}$ for some $z \in W$.

Proof: For any $T \subseteq S, w \in W$, then $w \in W_T = \langle T \rangle$ if and only if any geodesic (that is the shortest word) for $w$ is written solely in letters of $T$.

Suppose $xw = yw' \iff 1 = (xw)^{-1}yw' = w^{-1}x^{-1}yw' \iff 1 = yu'(xw)^{-1} = yw'w^{-1}x^{-1}$

Then we must have some cancelations.

(i) say $x^{-1}y$ cancels, ie. $x^{-1} = (x^*)^{-1}u^{-1}$, $y = uy^* \Rightarrow x = ux^*$ for some $u \in W$.

(ii) say $w'w^{-1}$ cancels, ie. $w' = (w')^*v$, $w^{-1} = v^{-1}(w^*)^{-1}$

$\Rightarrow w = w^*v$. Then $v \in W_{T \cap T'}$.

Suppose $u$ and $v$ are the longest words that cancel. Then we are left with $x^*w^*$ and $y^*(w')*$ to cancel. So these must be equal to 1. Thus $xw = u(x^*w^*)v = uv$ where $u \in W$ only depends on $x$ and $y$, so is the same for all $xw \in xW_T \cap yW_T$. Hence we have our result. \(\square\)

This behaviour is exactly our requirement for cells in a cell complex; that faces are again cells. Cohen \([6]\) uses this to construct a complex, which he calls a Chamber Complex for $W$, in which the cells are exactly cosets of the special subgroups. Krammer \([8]\) calls these cells precells, as they are the building blocks for the cells of the complex we want to construct: the Moussong Complex.

### 3.4 The Moussong Complex

Let $(W, S)$ be a Coxeter system, with finite rank.

The set of special subsets $T$ for which $W_T$ finite is itself a finite set, as it is a subset of the power set of $S$. Denote this set by $S$. From this point we will only consider these subsets.

Let $WS = \{wW_T : w \in W, T \in S\}$.
We construct simplices for the Moussong Complex $M(W)$, from chains of these cosets, as follows.

A $k$-simplex is given by: $w_0W_{T_0} \subset \ldots \subset w_kW_{T_k}$. Hence the vertices are of the form $w_0W_{T_0}$.

**Lemma 3.4.1** We can write a $k$-simplex as $wW_{T_0} \subset \ldots \subset wW_{T_k}$.

Proof: $w_i W_{T_i} \subset w_{i+1} W_{T_{i+1}} \iff w^{-1}_{i+1} w_i W_{T_i} \subset W_{T_{i+1}}$

$\Rightarrow w_{i+1} w_i 1 \in W_{T_{i+1}} \Rightarrow w_i \in w_{i+1} W_{T_{i+1}}$

So inductively, $w_i \in w_j W_{T_j}$ for each $i = 0, 1, \ldots, k$.

Note that $W_\emptyset = 1$, and $W_S = W$ if $W$ is finite.

$W$ acts on the cells (simplices) of $M(W)$ by left multiplication, as follows:

$w' (wW_{T_0} \subset \ldots \subset wW_{T_k}) = w' w W_{T_0} \subset \ldots \subset w' w W_{T_k}$.

Another way to describe $M(W)$ is by translating a "fundamental chamber".

Let $K$ be the simplicial complex with $k$-simplices of the form $W_{T_0} \subset \ldots \subset W_{T_k}$.

Then $M(W) = W \times K / \sim$, where

$(w_1, W_{T_0} \subset \ldots \subset W_{T_k}) \sim (w_2, W_{T_0} \subset \ldots \subset W_{T_k})$

$\iff w_1 W_{T_0} \subset \ldots \subset w_1 W_{T_k} = w_2 W_{T_0} \subset \ldots \subset w_2 W_{T_k}$

$\iff w^{-1}_1 w_2 \in W_{T_0}$.

This relation is generated by:

$(w, W_{T_0} \subset \ldots \subset W_{T_k}) \sim (ws, W_{T_0} \subset \ldots \subset ws W_{T_k})$

$\iff W_{T_0} \subset \ldots \subset W_{T_k} = s W_{T_0} \subset \ldots \subset s W_{T_k}$

$\iff w^{-1} ws = s \in W_{T_0}$.

So $M(W)$ can be thought of as being made up of copies of $K$ (translates of $K$ by the group elements of $W$). So $K$ is called the *fundamental chamber* of $M(W)$. We glue the faces of these pieces if the faces are both $k$-simplices whose first precells share a generator.

It is clear that the highest dimension of a cell of $M(W)$ will be $\max |T|$, since for a $k$-simplex the inclusions are strict, so each step you need to increase $|T|$ by at least 1 element. Thus the dimension of $M(W)$ is $\leq |S|$.

The *Stablizer* of a subset $X$ of the space $M(W)$ is the subset of group elements that fix $X$. That is, $\text{Stab}(X) = \{ w' \in W \mid w'X = X \}$.  

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Proposition 3.4.1 The points of $M(W)$ have finite stabilizers.

Proof: Each point of $M(W)$ lies in a cell. Each cell $wW_{T_0} \subset \ldots \subset wW_{T_k}$ of $M(W)$ has stabilizer $w'$ if $w'wW_{T_0} \subset \ldots \subset w'wW_{T_k} = wW_{T_0} \subset \ldots \subset wW_{T_k}$ which happens when $w'wW_{T_0} = wW_{T_0} \Leftrightarrow w^{-1}w'w \in W_{T_0} \Leftrightarrow w' \in wW_{T_0}w^{-1}$. Then since there are only finitely many of these special cosets and $w$ is fixed, then each cell has finite stabilizer. □

3.5 Cube Decomposition of $M(W)$

Let $K$ be the fundamental chamber of $M(W)$.

For any two special subsets $T_0, T_1 \in S$, define the interval $[T_0, T_1]$ to be the full subcomplex of $K$ generated by all the subsets $T \in S$ such that $T_0 \subset T \subset T_1$.

So if $T_1 = T_0 \cup \{s_1, \ldots, s_n\}$, then we have $n$-chains $W_{T_0} \subset W_{T_0 \cup \{s_i\}} \subset W_{T_0 \cup \{s_i, s_j\}} \subset \ldots \subset W_{T_1}$.

This is combinatorially an $n$-cube, since there are exactly $2^n$ ways of ordering in these chains.

For example in two and three dimensions:

![Figure 3.7](image)

The broken lines indicate the other chains that would be present in the simplicial form of $M(W)$. These chains contain “jumps” of more than one generator in the steps, so they are contained in higher dimensional cells (cubes).

A face of a cube given in this way is a restriction of the interval or subcomplex $[T_0, T_1]$ obtained by taking out a subset of $T_1 - T_0$.

The front face is given by suppressing the generator $r \in T_1 - T_0$ to give the interval $[\emptyset, \{s, t\}]$. 

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This corresponds exactly to our combinatorial description of a cube. Then $K$ can be decomposed into cubes instead of simplices, to make it a cubical complex. This induces a decomposition of $M(W)$ into cubes. **Example.** $M(W)$ for the dihedral group of order three:

**Proposition 3.5.1** $M(W)$ is a Euclidean cubical complex.

Proof: The above shows that $M(W)$ is combinatorially a cubical complex. To make it Euclidean, we need to put a Euclidean metric on the cells of $M(W)$ which will agree on adjacent faces. That is, we need the cubes to be isometric to a convex subspace of $\mathbb{E}^n$.

Each $(W_T, T)$ is a finite subsystem of $(W, S)$, so is a finite Coxeter system. Thus its reflection representation induces an action on $\mathbb{R}^{|T|}$. Let $|T| = n$. $W$ acts on real space $\mathbb{R}^{|T|} = \mathbb{R}^n$ by reflections; this action is generated by $\{\sigma_s | s \in T\}$. $e_s^\perp$ is the fixed point set for $\sigma_s$, and is a hyperplane perpendicular to the basis vector $e_s$. $e_s^\perp$ is codimension 1.

We can partition $\mathbb{R}^n$ into chambers by taking all the translates of the hyperplanes $e_s^\perp \forall s \in T$ by the reflections $\sigma_t$ for all $t \in T$. We call these hyperplanes walls.
Example. $D_3$

Note that the wall $\sigma_t e^\perp_s$ is the fixed point set of the reflection $\sigma_t \sigma_s \sigma_t$, since

$$\sigma_t \sigma_s \sigma_t (\sigma_t e^\perp_s) = \sigma_t (\sigma_s e^\perp_s) = \sigma_t (e^\perp_s).$$

Since we have a representation of a finite group, we will always get such a relation for every pair of generators. So the walls partition the space in a discrete way.

For each wall $e^\perp_s$ consider the closed half space of $\mathbb{R}^n$ on the “positive” side of the wall, that is, such that the coefficients for $e_s$ are positive or zero. The fundamental domain of this division is that given by the intersection of the closed half spaces of the hyperplanes $e^\perp_s \forall s \in T$. Denote it by $C$.

We want to choose a point of $C$, and look at its orbit under the action of the group. If we choose a point on a wall of $C$, then it is fixed by the action.
of the corresponding generator of $W_T$. Thus we need a point \textit{interior} to $C$. Let $x = e_1 + \ldots + e_n$. Then $x$ lies interior to $C$ at distance 1 from every wall $e_s^\perp$. The action of the reflection $\sigma_s$ merely changes the sign in front of the appropriate basis element, sending $x$ to the opposite side of the wall $e_s^\perp$ without altering its position with respect to the other walls. Thus $x$ has a discrete orbit under the action of the group $W_T$, since it is assumed finite.

Take the convex hull of the orbit of $x$ in $\mathbb{R}^n$, which we denote by $CH(W_T(x))$.

Figure 3.12:

Then by our original definition, $CH(W_T(x))$ is a (Euclidean) cell, being a convex hull of a set of points in $\mathbb{R}^n$. So since $C$ is the intersection of finitely many closed half spaces, intersecting it with a cell gives a cell, thus $CH(W_T(x)) \cap C$ is a Euclidean cell.

Further, we can show that it is combinatorially an $n$-cube. The point $x$ is given by co-ordinates $(1, 1, \ldots, 1)$ with respect to the given basis. The other vertices are obtained by placing zeroes in the co-ordinates, which corresponds to being on the intersection of some walls. More formally, a zero in the $i$-th co-ordinate means the point lies on the wall $e_i^\perp$. So the vertices of $CH(W_T(x)) \cap C$ are given by the set of maps from $I_{|T|}$ to $\{0, 1\}$, which makes it exactly a cube.

We will relabel the vertices of $CH(W_T(x)) \cap C$ in the following way. A vertex is labelled $W_T'$ if it is fixed by the reflections $\sigma_t \forall t \in T'$. Since the point $x$ does not lie on any wall of $K$, it is not fixed by any reflection, so we label it $W_\emptyset$. The origin is fixed by every reflection, so is labelled $W_T$.

In this way, it is clear that $CH(W_T(x)) \cap C$ is combinatorially equivalent to a cell of the fundamental chamber $K$. 50
This gives us a metric on each cell of $M(W)$, which must agree on faces since it is defined in the same way as the complex.

We can easily find the interior angles, since we are in Euclidean space. The angle between $e_i$ and $e_j$ is $\pi - \frac{\pi}{m_{ij}}$, therefore the angle at the origin between $e_i^\perp$ and $e_j^\perp$ is $\frac{\pi}{m_{ij}}$.

![Figure 3.13:](image)

We have right angles where the outer boundary of the cell meets the walls (since this is where the orbit of $x$ is reflected orthogonally in the walls), so since the sum of angles of a parallelogram in Euclidean space is $2\pi$, the angle at $x$ is $2\pi - \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{m_{ij}}\right) = \pi - \frac{\pi}{m_{ij}}$.

**Proposition 3.5.2** $W$ acts co-compactly discretely by isometries on $M(W)$.

Proof: We know that $M(W)$ factored by the action of $W$ is the fundamental chamber $K$. $K$ is compact, since it is constructed from chains of finite special cosets, of which there are a finite number. So we have co-compactness.

$W$ acts on $M(W)$ by translating the fundamental chamber, therefore $W$ acts by isometries of $M(W)$ since all the translates of the fundamental chamber are isometric copies of each other.

To show discreteness, suppose $w_1(x), w_2(x), \ldots$ are accumulating at a point $x_\infty$ in $M(W)$. Denote $x_i = w_i(x)$. Then $\lim_{i \to \infty} d(w_{i+1}(w_i^{-1}(x_i)), x_i) = 0$.

This means $w_{i+1}w_i^{-1}$ stabilizes $x_i$, but since there are only finitely many stabilizers for $M(W)$, there can only be a finite number of such $w_{i+1}w_i^{-1}$. Thus the accumulating sequence is finite, so the action is discrete.  

\[\square\]
We would like to show that $M(W)$ is simply connected, to apply the theorems from the previous chapter. The way we go about this is to show that a loop in $M(W)$ can deform onto the Cayley complex, which we know is simply connected, so the loop in $M(W)$ must be trivial. So we need to see how the Cayley complex fits into $M(W)$.

**Proposition 3.5.3** The Cayley complex embeds in $M(W)$.

Proof: We first show that the Cayley graph embeds in $M(W)$. To see it explicitly; an edge $(w, ws)$ of $\Gamma$ is the union of edges $wW_\emptyset \subset wW_{\{s\}}$ and $wsW_\emptyset \subset wsW_{\{s\}}$, where $wW_{\{s\}} \sim wsW_{\{s\}}$ in $M(W)$. Thus a vertex $w \in \Gamma$ corresponds to the vertex $wW_\emptyset$ of $M(W)$. For every pair of generators $s_i, s_j$ with $m_{ij} < \infty$, the interval $[\emptyset, \{s_i, s_j\}]$ is a 2-cube in $M(W)$, since $W_{\{s_i, s_j\}}$ is finite. Then translating this cell by the elements of $W_{\{s_i, s_j\}}$ and identifying adjacent edges appropriately we get a $2m_{ij}$-gon with centre labelled $W_{\{s_i, s_j\}}$, and boundary corresponding to the loop $1, s_i, s_j, s_i, \ldots, (s_i s_j)^{m_{ij}} = 1$ in the Cayley graph. A good example of this is Figure 3.9. Then in the Cayley complex, this loop is filled with a $2m_{ij}$-gon, which is what we have. Therefore the 2-dimensional subcomplex of $M(W)$ given by all cells $w[\emptyset, T]$ with $|T| \leq 2$ is a subdivision of the Cayley complex. So the Cayley complex embeds in $M(W)$. □

**Proposition 3.5.4** $M(W)$ is simply connected.

Proof: The fundamental chamber $K$ is contractible, since every cube is contained in a cube (cell) of the form $[\emptyset, T]$, so each cell is contractible to the point $W_\emptyset$. Then we can contract every translate of $K$ down to a 2-dimensional face containing translates of $W_\emptyset$.

Suppose $C$ is a closed loop in $M(W)$. Then in each cell through which it passes, it can be deformed down to a 2-dimensional face of the cell.

This deformation agrees on faces, so deforming the whole of $C$ to the 2-skeleton is well defined. Then since the 2-skeleton of $M(W)$ is a decomposition of the Cayley complex which is simply connected, then this loop is homotopically trivial. Hence our result. □

We need to show that $M(W)$ satisfies the Link Condition. The vertices of $M(W)$ are of the form $wW_T$. We will restrict ourselves to those of the form $W_T$, since the others are just translations of these.

Then any vertex $W_T$ where $T \neq \emptyset$ is interior to a subcomplex of $M(W)$ that has $W_T$ as a cone point, by construction. Metrically, the link around
such a vertex restricted to the subcomplex is a complete $|T|$-sphere, so is clearly CAT(1).

We say $T \in S$ is maximal in $S$ if $T$ is not contained in any other subset $T' \in S$.

If $T$ is maximal in $S$, then the link of $W_T$ is CAT(1), since it is a complete $T$-sphere. If $T$ is not maximal, then suppose $C$ is a closed geodesic in $LK(W_T, M(W))$. If we contract the higher dimensional cells in the star of $W_T$ onto the subcomplex containing $W_T$ as its cone point, then one can argue that the length of $C$ does not increase. Then in this subcomplex it is as if $T$ were maximal, so by the same argument as above, the link is CAT(1). Thus $C$ must have length at least $2\pi$. Thus the link of any interior vertex is CAT(1). For a more detailed proof, see Krammer [8].

Therefore to show that $M(W)$ satisfies the Link Condition, we are reduced to showing that $LK(W_\emptyset, M(W))$ is CAT(1). Since the neighborhood of $W_\emptyset$ is completely contained in the fundamental chamber $K$, we are reduced further to showing $LK(W_\emptyset, K)$ is CAT(1).

**Proposition 3.5.5** $LK(W_\emptyset, K)$ is a metric flag complex, and has edges length $\geq \frac{\pi}{2}$.

Proof: $u_i$ is a vertex of $LK(W_\emptyset, K)$ if $[\emptyset, \{s_i\}]$ is an edge of $K$.

$(u_i, u_j)$ is an edge of $LK(W_\emptyset, K)$ if there is a 2-cube $[\emptyset, \{s_i, s_j\}]$ of $K$. This means that $m_{ij}$ is finite.

Thus if $u_1, \ldots, u_n$ are pairwise joined vertices of $LK(W_\emptyset, K)$, then $m_{ij}$ is finite for all $1 \leq i, j \leq n$. So the $n$-cube $[\emptyset, \{s_1, \ldots, s_n\}]$ is in $K$, thus there is a simplex spanned by the vertices $u_1, \ldots, u_n$ in $LK(W_\emptyset, K)$.

Now an edge of the link corresponds to a 2-cube $[\emptyset, \{s_i, s_j\}]$ of $K$. We know from the isometry of this cell to Euclidean space using the reflection representation that the angle at the vertex $W_\emptyset$ made by the edges $[\emptyset, \{s_i\}], [\emptyset, \{s_j\}]$ which correspond to the walls $e_i^\perp$ and $e_j^\perp$ is $\pi - \frac{\pi}{m_{ij}} \geq \frac{\pi}{2}$.

Since the edge of the link has length equal to this angle, we have our result. \[\square\]

**Corollary 3.5.6** $M(W)$ satisfies the Link Condition.

Proof: We have shown that the link of vertices in the interior of $M(W)$ have no closed geodesics of length less than $2\pi$, which is equivalent to being CAT(1) by Proposition 2.4.2. Then since the link of $W_\emptyset$ in $K$ is a flag
complex with edges length $\geq \frac{\pi}{2}$, then this (and its translates) is also CAT(1), by Proposition 2.5.1. □

We are now ready to prove our main theorem.

**Theorem 3.5.7 (Moussong)**

Given a Coxeter System $(W, S)$, there is a CAT(0) cell complex on which $W$ acts co-compactly and discretely by isometries.

Proof: We have shown that the simplicial complex for $M(W)$ is acted on co-compactly and discretely by isometries by $W$, and this is isometric to the cubical complex. Further, $M(W)$ is a Euclidean cubical complex, and it satisfies the Link Condition. Hence by Theorem 2.4.1, $M(W)$ is locally CAT(0). Finally, since $M(W)$ is simply connected, then by the Cartan-Hadamard Theorem (1.7.8), $M(W)$ is a CAT(0) complex globally, and we have our result. □
Bibliography


