A NON-HOPFIAN ALMOST CONVEX GROUP

MURRAY J. ELDER

Abstract. In this article we prove that an isometric multiple HNN extension of a group satisfying the falsification by fellow traveler property is almost convex. As a corollary, Wise’s example of a CAT(0) non-Hopfian group is almost convex.

1. Introduction

In this article we give a proof that certain multiple HNN extensions enjoying certain geodesic conditions are almost convex. A non-Hopfian example of Wise [10] satisfies these conditions, so we have a non-Hopfian (hence non-residually finite) almost convex group.

The article is organized as follows. In Section 2 we define the properties of almost convexity and the falsification by fellow traveler property, and state some facts about them. We then define a multiple HNN extension and the properties of being strip equidistant and totally geodesic. A multiple HNN extension satisfying both conditions and having finitely generated free associated subgroups is said to be an isometric multiple HNN extension. In Section 3 we give two simple examples of isometric multiple HNN extensions, including Wise’s non-Hopfian example. In Section 4 we state and prove the main theorem about almost convexity for isometric multiple HNN extensions.

The author wishes to thank Walter Neumann, Jon McCammond and Mike Shapiro for their help, and the article’s referee for her/his careful reading and useful comments.

2. Definitions

Let $G$ be a group with finite generating set $X$, and let $\Gamma(G,X)$ be the corresponding Cayley graph.

Definition 2.1 (Almost convex). $(G,X)$ is almost convex if there is a constant $C > 0$ such that for every pair of elements $g,g'$ in the metric sphere of radius $N$ at most distance 2 apart in $\Gamma(G,X)$ there is a path of length at most $C$ from $g$ to $g'$ which runs inside the metric ball of radius $N$.

Cannon introduced the notion of an almost convex group in [1], where he proved that if a pair $(G,X)$ is almost convex then there is an efficient algorithm to construct any finite portion of its Cayley graph. Word hyperbolic groups, Coxeter groups and the fundamental groups of closed 3-manifolds with one of the eight geometries except Solvgeometry are almost convex [2, 8].

2000 Mathematics Subject Classification. 20F65.
Key words and phrases. almost convex, non-Hopfian, falsification by fellow traveler property.
Neumann and Shapiro gave a definition which they attributed to Cannon, of a nice property that turns out to be relevant to almost convex groups.

**Definition 2.2** (The falsification by fellow traveler property). $(G, X)$ has the (asynchronous) falsification by fellow traveler property if there is a constant $k > 0$ such that every non-geodesic word is (asynchronously) $k$-fellow traveled by a shorter word in $\Gamma(G, X)$.

If a pair $(G, X)$ enjoys the asynchronous falsification by fellow traveler property then it also enjoys the synchronous falsification by fellow traveler property [3]. If $(G, X)$ has the falsification by fellow traveler property then the full language of geodesics on $X$ is regular [7].

**Proposition 2.3** (Falsification by fellow traveling implies almost convex). If $(G, X)$ has the falsification by fellow traveler property then $(G, X)$ is almost convex.

The proof of this can be found in [3]. If $k$ is the falsification by fellow traveler property constant then the almost convex constant $C$ is at most $3k$. The converse of Proposition 2.3 is false; in the present article we prove that Wise’s example is almost convex, and in [4] we prove that the full language of geodesics is not regular for the same generating set, hence it doesn’t enjoy the falsification by fellow traveler property.

The enjoyment of either property is dependent on the choice of generating set [7, 9]. The following result is proved in [7], where the authors go on to prove that any virtually abelian group has a generating set for which it enjoys the falsification by fellow traveler property.

**Proposition 2.4** (Abelian implies the falsification by fellow traveler property). Any finite generating set for an abelian group has the falsification by fellow traveler property.

**Definition 2.5** (Multiple HNN extension). Let $A$ be a group with finite generating set $X$ and relators $R$. Define an isomorphism $\phi_i : U_i \to V_i$ between pairs of isomorphic subgroups $U_i, V_i$, for $i \in [1, n]$. The multiple HNN extension of $(A, X)$ with these isomorphisms is the group $G$ with presentation

$$\langle X, s_1, \ldots, s_n | R, s_i^{-1}u_is_i = \phi_i(u_i); u_i \in U_i, i \in [1, n] \rangle.$$

The generators $s_i$ are called **stable letters**, and a subword of the form $s_i^{-1}u_is_i$ or $s_iv_is_i^{-1}$ is called a pinch, where $u_i \in U_i, v_i \in V_i$. A word that contains no pinches is called **stable letter reduced**. Britton’s Lemma states that if a freely reduced word containing stable letters is non-trivial and represents the identity in the group then it must contain a pinch. Two words are said to have **parallel stable letter structure** if they have the exact same sequence of stable letters (when we ignore the elements of the base group). See [6] for more details about HNN extensions.

When each $U_i$ is finitely generated, that is, $U_i = \{u_{ij} : j = 1, \ldots, m_i\}$, then define $v_{ij} = \phi_i(u_{ij})$, and since $\phi_i$ is an isomorphism $V_i$ is generated by $\{v_{ij} : j = 1, \ldots, m_i\}$, and the multiple HNN extension has the presentation

$$\langle X, s_1, \ldots, s_n | R, s_i^{-1}v_is_i = v_{ij}; i \in [1, n], j \in [1, m_i] \rangle,$$

which is finite when $R$ is finite.

If $X$ is an alphabet let $X^*$ denote the set of all words in the letters of $X$ (including the empty word).
Figure 1. The presentation 2-complex for \((G_W, \{a, b, c, s, t\})\).

**Definition 2.6** (Geodesic, totally geodesic, strip equidistant). We say associated subgroups are *geodesic* if each freely reduced word in \(\{u_{i1}^{\pm1}, \ldots, u_{im_i}^{\pm1}\}^*\) and \(\{v_{i1}^{\pm1}, \ldots, v_{im_i}^{\pm1}\}^*\) is geodesic, and *totally geodesic* if for each geodesic word \(w \in U_i\) [respectively \(V_i\)], \(w \in \{u_{i1}^{\pm1}, \ldots, u_{im_i}^{\pm1}\}^*\) [respectively \(w \in \{v_{i1}^{\pm1}, \ldots, v_{im_i}^{\pm1}\}^*\)]. Note that totally geodesic subgroups are geodesic. We say the geodesic associated subgroups are *strip equidistant* if \(|u_{ij}| = |v_{ij}|\) for each \(i, j\). Finally, we say a presentation for a multiple HNN extension is (totally) geodesic [respectively strip equidistant] if all associated subgroups are.

**Definition 2.7** (Isometric multiple HNN extension). Let \(A = \langle X \mid R \rangle\) be a finitely presented group with pairs of isomorphic finitely generated free subgroups \(U_i, V_i\) for \(i \in [1, n]\). If the multiple HNN extension of \((A, X)\) associating these subgroups is strip equidistant and totally geodesic, then we call it an *isometric multiple HNN extension*.

The Cayley graph of such a presentation consists of copies of the Cayley graph of the base group \((A, X)\), glued together along the subspaces corresponding to the free subgroups \(U_i\) and \(V_i\) by stable letter “strips”. See Figure 2 in the next section for an illustration.

3. Examples

In this section we give two examples of groups in the class of isometric multiple HNN extensions.

**Example 1** (Wise). Let \(A = \mathbb{Z}^2 \cong \langle a, b, c, d \mid c = ab, c = ba, d = c^2 \rangle\) and define the multiple HNN extension \(G_W\) by associating pairs of cyclic subgroups \(\langle a \rangle, \langle d \rangle\) and \(\langle b \rangle, \langle d \rangle\). \(G_W\) has the presentation

\[
\langle a, b, c, d, s, t \mid c = ab, c = ba, d = c^2, s^{-1}as = d, t^{-1}bt = d \rangle.
\]

Wise showed that this group is non-Hopfian and CAT(0) \(^1\).

The Cayley graph for \((G_W, \{a, b, c, d, s, t\})\) is the universal cover of the 1-skeleton of the 2-complex shown Figure 1. It is easy to check that this 2-complex, metrized so that the edge labeled \(c\) has half the length of the other edges, satisfies the link condition so is CAT(0). This is the presentation that Wise used, minus the relation \(d = c^2\), to prove CAT(0). Note that with this metric, the triangle corresponding to this relation is degenerate, so looks a little strange in the figure. The Cayley

---

\(^1\)Wise claimed in [10] that this group is automatic; the proof given was incorrect and its automaticity is as yet unresolved.
A NON-HOPFIAN ALMOST CONVEX GROUP

A graph can be viewed as being made up of “planes”, corresponding to copies of the Cayley graph of the base group \((A, X)\), in this case \((\mathbb{Z}^2, \{a, b, c, d\})\), glued together by “strips” made up of copies of the squares (more generally metric rectangles) in Figure 1 shown in Figure 2.

![Figure 2. A strip in \(\Gamma(G_W, \{a, b, c, d, s, t\})\)](image)

The presentation is by design strip equidistant, and totally geodesic, since the words \(a^n, b^n\) and \(d^n\) are unique geodesic representatives in \((\mathbb{Z}^2, \{a, b, c, d\})\) for elements of the associated subgroups \(\langle a \rangle, \langle b \rangle\) and \(\langle d \rangle\) respectively.

It follows that if a path in the Cayley graph is not stable letter reduced, then it can be shortened and therefore is not geodesic. Moreover two geodesics from the identity to the same plane have parallel stable letter structure, using Britton’s Lemma. In [4] we consider in detail the geodesic structure of this presentation.

**Example 2.** Let \(A = F_2 \cong \langle a, b, | - \rangle\) and define the (single) HNN extension \(G_2\) by defining an isomorphism between free subgroups \(\langle a^2, b^3 | - \rangle\) and \(\langle b^2, aba | - \rangle\) by \(\phi(a^2) = b^2, \phi(b^3) = aba\). \(G_2\) has the presentation \(\langle a, b, s | s^{-1}a^2s = b^2, s^{-1}b^3s = aba \rangle\).

The presentation 2-complex shown in Figure 3 is easily seen to satisfy the link condition, so is CAT(0). Moreover it can be viewed as a CAT(0) squared complex, so by Gersten-Short [5] is biautomatic.

It is easily checked that the associated subgroups are totally geodesic, and that the presentation is strip equidistant, so \((G_2, \{a, b, s\})\) is an isometric (multiple) HNN extension.

4. THE MAIN THEOREM

We now prove that an isometric multiple HNN extension is almost convex.

**Theorem 4.1.** Let \(\langle G, X \cup \{s_i\}_{i=1}^n \rangle\) be an isometric multiple HNN extension with base group \((A, X)\). If \((A, X)\) has the falsification by fellow traveler property then \(G\) is almost convex with respect to the generating set \(X \cup \{s_i\}_{i=1}^n\).

![Figure 3. The presentation 2-complex for \(G_2\)](image)
Proof. Let \( S(N) \) denote the metric sphere of radius \( N \) and \( B(N) \) the metric ball of radius \( N \) in \( \Gamma(G, X \cup \{s_i\}_{i=1}^n) \). Let \( g, g' \in S(N) \) with \( d(g, g') \leq 2 \) realized by a path \( \gamma \). Let \( w, w' \) be geodesic words for \( g, g' \) respectively. Since the presentation is strip equidistant, \( w \) and \( w' \) are stable letter reduced.

Note that the metric sphere [ball] of radius \( N \) in the base group \((A, X)\) is a subset of the sphere [ball] of radius \( N \) in \( \Gamma(G, X \cup \{s_i\}_{i=1}^n) \). Without loss of generality we may assume the falsification by fellow traveler property constant \( k \) for \((A, X)\) is an even integer. We may also assume that \( X \) is inverse closed. Denote the endpoint of a path \( w \) from the identity by \( \overline{w} \).

The argument is divided into 3 cases.

**Case 1.** \( w\gamma(w')^{-1} \) has no stable letters. Then since \((A, X)\) has the falsification by fellow traveler property we are done. (Recall that \( C = 3k \).)

**Case 2.** \( \gamma \) involves a stable letter. That is, \( \gamma = s, sx \) or \( xs \) for \( s \in \{s_i^{\pm 1}\}_{i=1}^n \) and \( x \) any generator (or inverse of a generator) except \( s^{-1} \). By Britton’s Lemma \( w\gamma(w')^{-1} \) contains a pinch so either \( w \) or \( w' \) has an \( s^{-1} \). Without loss of generality assume \( w = w_1s^{-1}w_2 \) where \( s^{-1}w_2s \) is a pinch.

If \( \gamma = s \) or \( \gamma = sx \) then the path \( w_1s^{-1}w_2 \) is shown in Figure 4(a). The subword \( w_2 \) is geodesic, and since \( s^{-1}w_2s \) is a pinch, it is an element of an associated subgroup \( U_i \) or \( V_i \). Since the presentation is totally geodesic, this path must run along the top of a strip, and the bottom of the strip is a word of the same length. Thus the point labeled \(*\) (which is \( \overline{w_2} \)) lies in \( B(N-1) \), so if \( \gamma = sx \) then \( \gamma \) lies in \( B(N) \) and if \( \gamma = s \) we have a contradiction.

![Figure 4](https://example.com/image.png)

**Figure 4.** Case 2
If $\gamma = xs$ then $w_2x$ evaluates to an element of an associated subgroup. If $x$ is a stable letter then we are in the previous case. So we may assume $w_2x \in X^*$. If $w_2x$ is geodesic then it must run along the strip, so without loss of generality there is some $u_{ij}$ ending in $x$, with $s^{-1}u_{ij}x = v_{ij}$. Let $u_{ij} = u'x$ and $w_2 = w_3u'$. We show the path $w_1s^{-1}w_3u'$ in Figure 4(b), and draw the subword $w_2x = w_3u_{ij} = w_3u'x$ on top of a strip. Consider the path $(u')^{-1}s_{ij}$ between $g$ and $g'$. The point labeled $\ast$ lies on the path $w_2x$ at distance $N - |u'|$ from the identity. The point labeled $\dagger$ is distance $N - |w_2| - 1 + |w_3| = N - |u'| - 1$ from the identity, since it can be reached by traveling along $w_1$ then along the bottom of the strip shown in Figure 4(b). Thus the path $(u')^{-1}s_{ij}$ between $g$ and $g'$ stays inside $B(N)$. It has length at most $2\max\{|u_{ij}| : u_{ij}$ is a generator of $U_i, i \in [1, n]\}$.

If $w_2x$ is not geodesic then let $y$ be the geodesic for $w_2x$ as in Figure 4(c). Now $|y| < |w_2x| = |w_2| + 1$ so $|y| \leq |w_2|$. There is a path $w_1sy$ to the point labeled $\ast$ in Figure 4(c) of length $|w_1| + 1 + |y| \leq |w_1| + 1 + |w_2| = N$ so if the point $\ast \in S(N)$ we have Case 2 (with $\gamma = s$, if $\ast \in S(N - 1)$ then $\gamma$ lies in $B(N)$ and if $\ast \in B(N - 2)$ then we have a contradiction.

**Case 3.** $\gamma$ has no stable letters and $w$ has a stable letters. Then $w, w'$ have parallel stable letter structure. Suppose $s \in \{s_{ij}\}_{i=1}^{\pm 1}$ is the last stable letter of $w$, so $w = w_1sw_2, w' = w'_1s'w'_2$, where $w_2$ and $w'_2$ have no stable letters.

We will draw the Cayley graph which contains the paths $w_2, \gamma$ and $w'_2$ in Figures 5 and 6. The path $w_2\gamma(w'_2)^{-1}$ starting at the point labeled $\ast$ in both figures is a word in $X^*$, and moreover evaluates to an element of an associated subgroup. Therefore it evaluates to a word that runs along the top of a strip. This strip is shown in each figure, and below it there are two paths (not shown) $w_1$ and $w'_1$ back to the identity.

If $w_2\gamma(w'_2)^{-1}$ is geodesic then it runs along the top of the strip. The path $\gamma$ could lie on top of at most two rectangles in the strip, so by tracing around these rectangles we stay within $B(N)$. Therefore we can find a path inside $B(N)$ from $g$ to $g'$ of length at most $4\max\{|u_{ij}| : u_{ij}$ is a generator of $U_i, i \in [1, n]\}$.

If $w_2\gamma(w'_2)^{-1}$ is not geodesic then by the falsification by fellow traveler property in $(A, X)$ there is a shorter word $u$ which synchronously $k$-fellow travels it. If $u$ is geodesic, it runs along the strip, and we show these paths in Figure 4. Let $y = sus^{-1}$

**Figure 5.** Case 3: $u$ is geodesic.

be the geodesic on the other side of the strip from $u$. Then $y$ has length at most
\(|w_2\gamma(w'_2)^{-1}| - 1\), starts at \(\overline{w_1T} \in S(N - |w_2| - 1)\) and ends at \(\overline{w'_1} \in S(N - |w'_2| - 1)\), so lies in \(B(N)\).

Let \(w(t)\) denote the point at distance \(t \in \mathbb{R}_{\geq 0}\) along the path \(w\) from its start point, where \(w(t) = \overline{w}\) for \(t \geq |w|\).

If \(|w_2|, |w'_2| \geq \frac{k}{2}\) then consider the path starting at \(\overline{w} = w(N)\) retracing along \(w\) to \(w(N - \frac{k}{2})\). The paths \(w_2\gamma(w'_2)^{-1}\) and \(u\) based at \(\ast\) in Figure 5 \(k\)-fellow travel in the base group \((A, X)\), so the paths \(w_1sw_2\gamma(w'_2)^{-1}\) and \(w_1su\) (based at the identity) \(k\)-fellow travel is the larger group. Thus there is a path of length at most \(k\) from \(w(N - \frac{k}{2})\) to \(w_1swu(N - \frac{k}{2})\) which lies in \(B(N)\). This path is drawn as a jagged line in the figure. From here cross to \(w_1y(N - \frac{k}{2})\) by an edge \(s^{-1}\), and travel along \(y\) (which is inside \(B(N)\)) to \(w_1y(N + |\gamma| + \frac{k}{2})\), then cross the strip by an edge \(s\) to \(w_1su(N + |\gamma| + \frac{k}{2})\). Again since \(w_1sw_2\gamma(w'_2)^{-1}\) and \(w_1su\) \(k\)-fellow travel there is a path of length at most \(k\) to \(w_1sw_2\gamma(w'_2)^{-1}(N + |\gamma| + \frac{k}{2})\). Now \(w_1sw_2\gamma(w'_2)^{-1}(N + |\gamma| + \frac{k}{2}) = w'(N - \frac{k}{2}) \in S(N - \frac{k}{2})\) and \(w_1su(N + |\gamma| + \frac{k}{2}) = w'_1su^{-1}(N - \frac{k}{2} - c)\) for some positive \(c\) since \(u\) is shorter than \(w_2\gamma(w'_2)^{-1}\), so the path between these points lies in \(B(N)\). Finally, travel along \(w'\) to \(\overline{w'}\). The concatenation of these paths stays within \(B(N)\), and has length at most \(\frac{k}{2} + k(2\frac{k}{2} + 2) + k + \frac{k}{2} = 4k + 4\).

If \(|w_2|, |w'_2| < \frac{k}{2}\) then we have a path from \(\overline{w}_1\) to \(\overline{w}_1'\), along \(y\) to \(\overline{w}_1'\), then along \(w'\) to \(\overline{w'}\), which runs inside \(B(N)\) and has length less than \(\frac{k}{2} + 1 + (2\frac{k}{2} + 2) + 1 + \frac{k}{2} = 2k + 4\).

If \(|w_2| < \frac{k}{2}, |w'_2| \geq \frac{k}{2}\) (or vice versa) then a combination of the above arguments gives a path inside \(B(N)\) of length at most \(\frac{k}{2} + 1 + (2\frac{k}{2} + 2) + 1 + k + \frac{k}{2} = 3k + 4\). Namely, travel back along \(w\) to \(\overline{w}_1\), then along \(y\) to \(w_1y(N + |\gamma| + \frac{k}{2})\), across the strip to \(w_1u(N + |\gamma| + \frac{k}{2})\) and by a path of length at most \(k\) to \(w_1sw_2\gamma(w'_2)^{-1}(N + |\gamma| + \frac{k}{2}) = w'(N - \frac{k}{2})\), then along \(w'\) to \(\overline{w'}\). The other case is similar.

If \(u\) is not geodesic then by the falsification by fellow traveler property there is a shorter word \(v\) in the base group \((A, X)\) which synchronously \(k\)-fellow travels \(u\) as shown in Figure 6. Then \(|v| \leq |w_2\gamma(w'_2)^{-1}| - 2 \leq |w_2| + |w'_2|\) so \(v\) lies in \(B(N)\).

![Figure 6](image-url)

**Figure 6.** Case 3: \(u\) is not geodesic.
here travel along $v$ (which lies in $B(N)$) to $w_1sv(N + |\gamma| + \frac{k}{2})$. From here there is a path of length at most $k$ to $w_1sw(N + |\gamma| + \frac{k}{2})$ and another to $w_1sw_2\gamma(w'_2)^{-1}(N + |\gamma| + \frac{k}{2})$. The point $w_1sv(N + |\gamma| + \frac{k}{2}) = w'_1sv^{-1}(N - \frac{k}{2} - c)$ for some positive $c$ and $w_1su(N + |\gamma| + \frac{k}{2}) = w'_1su^{-1}(N - \frac{k}{2} - d)$ for some positive $d$, so these paths lie in $B(N)$. Finally travel from $w_1sw_2\gamma(w'_2)^{-1}(N + |\gamma| + \frac{k}{2}) = w'(N - \frac{k}{2})$ back to $\overline{w'}$. The total length of the entire path is at most $\frac{k}{2} + k + (2\frac{k}{2} + 2) + k + k + \frac{k}{2} = 6k + 2$.

If $|w_2|, |w'_2| < \frac{k}{2}$ then $|v| \leq |w_2| + |w'_2| < k$ so there is a path from $\overline{w}$ to $\overline{w_1}$, then along $v$ to $\overline{w'_1}$, which runs inside $B(N)$ and has length less than $2k$.

If $|w_2| < \frac{k}{2}, |w'_2| \geq \frac{k}{2}$ (or vice versa) then a combination of the previous two arguments gives a path from $\overline{w}$ to $\overline{w'_1}$, then along $v$ to $w_1v(N + |\gamma| + \frac{k}{2})$. From here there is a path to $w_1u(N + |\gamma| + \frac{k}{2})$ of length at most $k$ which lies in $B(N)$, then a path of length at most $k$ to $w_1sw_2\gamma(w'_2)^{-1}(N + |\gamma| + \frac{k}{2}) = w'(N - \frac{k}{2})$, then along $w'$ to $\overline{w'}$, and its total length is at most $\frac{k}{2} + (2\frac{k}{2} + 2) + k + k + \frac{k}{2} = 4k + 2$. The other case is similar.

This completes all possible cases. The almost convexity constant $C$ is at most $\max\{6k + 2, |u_{ij}| : u_{ij} \text{ is a generator of } U_i, i \in [1, n]\}$ where $k$ is the falsification by fellow traveler property constant for $(A, X)$.

\begin{corollary}
Wise’s example is almost convex and non-Hopfian.
\end{corollary}

\begin{proof}
We showed in Example 1 that Wise’s example is an isometric multiple HNN extension, and the base group is abelian so by Proposition 2.4 satisfies the falsification by fellow traveler property. Wise proved that the example is non-Hopfian in [10].
\end{proof}

\begin{thebibliography}{10}
\footnotesize
\end{thebibliography}

\text{(AS AT DEC 2006) DEPT OF MATHEMATICAL SCIENCES, STEVENS INSTITUTE OF TECHNOLOGY, HOBOKE NJ 07030 USA)
E-mail address: meider@stevens.edu}